Exceptional Hsiang algebras and Steiner triple systems

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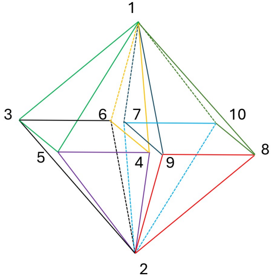

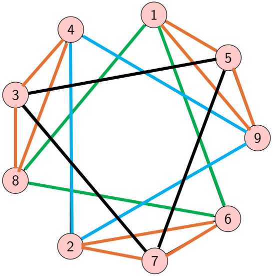

(a joint work with Daniel Fox, Universidad Politécnica de Madrid)

SNAG, Karlskrona, March 28th, 2025

The context: three equivalent problems

1. (Hsiang, 1967) Classify all **zero mean curvature** cones given by a homogeneous polynomial degree three equation $u(x) = 0$ in $\mathbb{R}^n$.
2. To find all homogeneous degree three polynomial solutions of
$$\tfrac{1}{2}\nabla u(x) \cdot \nabla(|\nabla u(x)|^2) - |\nabla u|^2 \Delta u = \theta \langle x; x \rangle u(x)$$
3. (V.T.) Classify commutative nonassociative metrized algebras on $\mathbb{R}^n$ with
$$\begin{aligned} &4x(x(xx)) + (xx)(xx) - 2\theta\langle x; x\rangle(xx) - \tfrac{4}{3}\theta\langle xx; x\rangle x = 0, \\ &\operatorname{tr} L_x = 0. \end{aligned} \tag{1}$$

Definition

A commutative algebra with an invariant form (i.e. $\langle xy; z\rangle = \langle x; zy\rangle$) satisfying (1) is called a Hsiang algebra.

A correspondence between (2) and (3):

$$\boxed{u(x) = \tfrac{1}{6}\langle x; x^2\rangle \qquad \leftrightarrows \qquad \text{a 'composition law'} \quad x\,y = \operatorname{Hess} u(x) y}$$

A toy example: several faces of $\mathbb{E}_2$

A **solution of the PDE** in $\mathbb{R}^2$ is given by $u(x_1, x_2) = \frac{1}{6}(x_1^3 - 3x_2^2 x_1)$ (describes a minimal Steiner tree in $\mathbb{R}^2$). Then

$$\operatorname{Hess} u(x) = \begin{pmatrix} x_1 & -x_2 \\ -x_2 & -x_1 \end{pmatrix} =: L(x), \qquad \operatorname{tr} L(x) = 0, \tag{2}$$

$$(x_1, x_2)\,(y_1, y_2) = (x_1 y_1 - x_2 y_2, -x_2 y_1 - x_1 y_2) = \overline{(x_1, x_2) \cdot (y_1, y_2)},$$

where $\cdot$ is the complex number multiplication on $\mathbb{R}^2 \cong \mathbb{C}$, i.e. $\mathbb{A}(u)$ is the algebra of **para-complex numbers**. The bilinear form $\langle x; y\rangle := \operatorname{Re}(\bar{x} \cdot y)$ is invariant:

$$\langle xy; z\rangle := \operatorname{Re}(\overline{(\bar{x} \cdot \bar{y})} \cdot z) = \operatorname{Re}(x \cdot y \cdot z) = \langle x; zy\rangle.$$

$\mathbb{A}(u) \cong \widehat{\mathbb{C}} \cong \mathbb{E}_2$ (the **Harada** or a **simplicial** algebra) and

$$xx = \bar{x} \cdot \bar{x} = \bar{x}^{\cdot 2}, \qquad x(xx) = \bar{x} \cdot \bar{\bar{x}} \cdot \bar{\bar{x}} = |x|^2 x$$

$$x(x(xx)) = |x|^2 \bar{x}^{\cdot 2}, \qquad (xx)(xx) = x^{\cdot 4}$$

therefore satisfies the **Hsiang identity** for $\theta = \frac{3}{2}$:

$$\begin{aligned} &4x(x(xx)) + (xx)(xx) - 2\theta\langle x; x\rangle(xx) - \tfrac{4}{3}\theta\langle xx; x\rangle x \\ &= 4|x|^2\bar{x}^{\cdot 2} + x^{\cdot 4} - 3|x|^2 x^{\cdot 2} - (x^{\cdot 3} + \bar{x}^{\cdot 3}) \cdot x = 0. \end{aligned}$$

A cubic form u on a $\mathbb{K}$-vector space V is a map $V \to \mathbb{K}$ which is homogeneous of degree 3 and which extends to arbitrary scalar extensions $\mathbb{K}_\Omega$ by

$$u(\sum_i \omega_i x_i) = \sum_i \omega_i^3 u(x_i) + \sum_{i \neq j} \omega_i^2 \omega_j u(x_i; x_j) + \sum_{i<j<k} \omega_i \omega_j \omega_k u(x_i; x_j; x_k)$$

where $u(x; y)$ is quadratic in x, linear in y, $u(x; y; z)$ is symmetric and trilinear.

Definition

An algebra $\mathbb{A}$ with a non-degenerate **invariant** symmetric bilinear form $\langle ; \rangle$ is called **metrized**. Given metrized algebra, we define the **defining form** $u(x) = \frac{1}{6}\langle xx; x\rangle$.

An important observation

The multiplication $(x, y) \to xy$ is uniquely determined by the defining form:

$$u(x; y; z) = h(xy, z) \qquad \text{for any } z \in \mathbb{A}.$$

Equivalently, the structure constants a_{ijk} in any ON-basis $\{e_i\}$ of $\mathbb{A}$ are defined by

$$e_i e_j = e_j e_i = \sum_k a_{ijk} e_k, \qquad \text{where} \quad u(x) = \sum_{1 \le i,j,k} a_{ijk} x_i x_j x_k. \tag{3}$$

If c is an idempotent in a commutative **metrized** algebra $\mathbb{A}$ then $L(c)$ is **diagonalizable**: there exists an orthogonal ('Peirce') decomposition relative to c

$$\mathbb{A} = \bigoplus_i \mathbb{A}_c(\lambda_i), \quad L(c) = \lambda_i \,\mathrm{id}_{\mathbb{A}_c(\lambda_i)}, \quad 1 \le i \le s.$$

An important tool is the fusion laws $(i,j) \to i \star j \in 2^{\mathbb{N}_s}$ between the blocks:

$$\mathbb{A}_c(\lambda_i)\mathbb{A}_c(\lambda_j) \subset \bigoplus_{k \in i\star j} \mathbb{A}_c(\lambda_k).$$

Any Hsiang algebra satisfy the following fusion laws
(the **hidden Clifford structure** highlighted):

$\star$	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
1	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
-1	-1	1	$\frac{1}{2}$	$-\frac{1}{2} \oplus \frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$1 \oplus -\frac{1}{2}$	$-1 \oplus \frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2} \oplus \frac{1}{2}$	$-1 \oplus -\frac{1}{2}$	$1 \oplus -1 \oplus -\frac{1}{2}$

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Any Hsiang algebra satisfy the following fusion laws
(the **hidden Jordan structure** highlighted):

$\star$	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
1	1	-1	$-\frac{1}{2}$	$\frac{1}{2}$
-1	-1	1	$\frac{1}{2}$	$-\frac{1}{2} \oplus \frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$1 \oplus -\frac{1}{2}$	$-1 \oplus \frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2} \oplus \frac{1}{2}$	$-1 \oplus -\frac{1}{2}$	$1 \oplus -1 \oplus -\frac{1}{2}$

Example 1

Pseudo-composition algebras (Meyberg, Osborn, Walcher etc):

$$x^3 = h(x, x)x$$

with an invariant bilinear form h. Under an additional assumption $\operatorname{tr} L(x) = 0$, any pseudo-composition algebra is Hsiang.

Example 2

Elduque and Okubo (2000) studied commutative algebras satisfying

$$x^2 x^2 = N(x)x$$

and proved an existence of an invariant bilinear form h such that $N(x) = h(x^2, x)$. Again, under an additional assumption $\operatorname{tr} L(x) = 0$, any such an algebra is Hsiang.

Example 3

Given a simple cubic Jordan algebra $\mathbb{A}$ and its subalgebra $\mathbb{B}$, the contraction of the algebra structure onto $\mathbb{B}^{\perp}$ (with respect to the generic trace form of $\mathbb{A}$) is a Hsiang algebra.

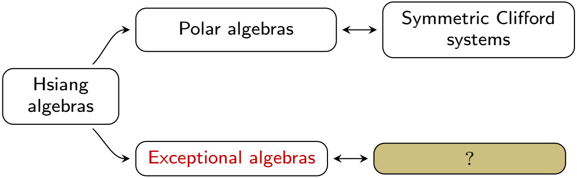
Polar vs exceptional algebras

A metrized $\mathbb{Z}_2$-graded algebra $\mathbb{A} = \mathbb{A}_0 \oplus \mathbb{A}_1$ is said to be **polar** if $\mathbb{A}_0\mathbb{A}_0 = \{0\}$ and

$$x_0(x_0x_1) = h(x_0, x_0)x_1, \qquad x_0 \in \mathbb{A}_0, x_1 \in \mathbb{A}_1.$$

Theorem (V.T. 2010)

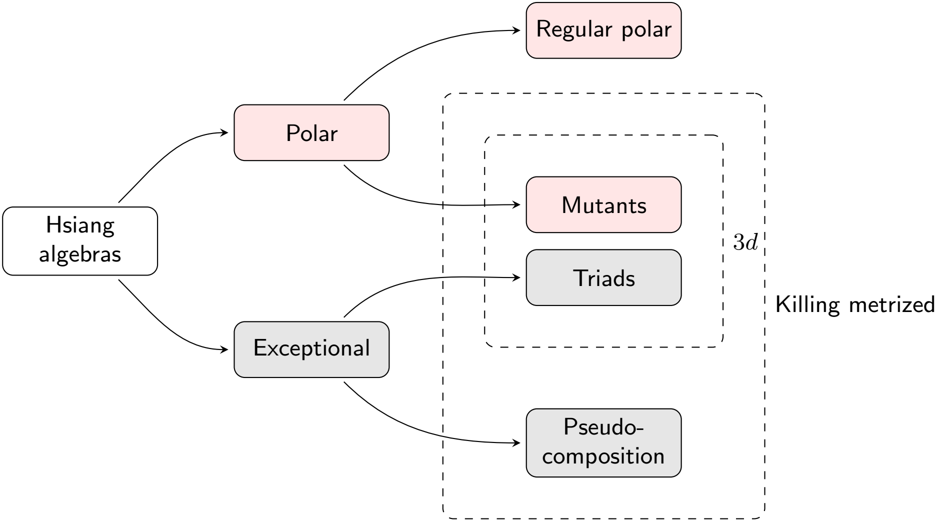
Any polar algebra is canonically associated with a symmetric Clifford system (this yields an effective classification). Polar algebras exist **in almost all dimensions**.



Theorem (Nadirashvili, Vladuts, V.T. 2014)

There are **only finitely many dimensions** n where exceptional algebras can exist.

Mutants


Regular polar
Polar
Hsiang algebras
Mutants
3d
Triads
Killing metrized
Exceptional
Pseudo-composition

Mutants

A borderline case of polar algebras, called mutants, share important properties of exceptional algebras. Given a unital composition algebra $\mathbf{H}_d$ over $\mathbb{K}$, with unity e, conjugation $\bar{x}$, norm $n(x)$, $\dim_{\mathbb{K}} \mathbf{H}_d = d \in \{1, 2, 4, 8\}$, a **mutant** is the **tripling**

$$\mathrm{Tri}(\mathbf{H}_d) := \mathbf{H}_d \times \mathbf{H}_d \times \mathbf{H}_d = V_1 \oplus V_2 \oplus V_3$$

with **commutative** multiplication

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = (\bar{x}_3\bar{y}_2 + \bar{y}_3\bar{x}_2, \bar{x}_1\bar{y}_3 + \bar{y}_1\bar{x}_3, \bar{x}_2\bar{y}_1 + \bar{y}_2\bar{x}_1)$$

and an invariant bilinear form $H((x_1, x_2, x_3), (y_1, y_2, y_3)) = \sum_{i=1}^{3} t(\bar{x}_i y_j)$, where $t(x) = n(x + e) - n(x) - n(e)$ is the trace form ('the real part'). Note that

$$V_i V_i = 0, \qquad V_i V_j = V_k$$

(cf. the concept of Cartan's triality)

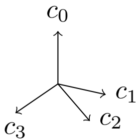
Proposition (D. Fox, V.T., 2024)

$\mathrm{Tri}(\mathbf{H}_d)$ is a **polar algebra** w.r.t. to any of the three decompositions $V_k \oplus V_k^{\perp}$. The corresponding defining form $u(x) = u(x_1, x_2, x_3) = t(x_1(x_2x_3))$.

The **simplicial** algebra $\mathbb{E}_3$

$\mathbb{E}_3$ is the 3-dimensional algebra over $\mathbb{K}$ generated by four **idempotents** c_i, $0 \le i \le 3$, subject to the conditions:

$$(c_i + c_j)^2 = 0 \quad \Leftrightarrow \quad c_i c_j = -\tfrac{1}{2}(c_i + c_j), \quad i \neq j.$$



Proposition

1. $\mathbb{E}_3$ is a Hsiang algebra (a mutant), metrized with respect to the natural (Killing) form $h(x,y) := \operatorname{tr} L(x)L(y)$ with the Peirce decomposition
$$\mathbb{E}_3 = \mathbb{A}_{c_i}(1) \oplus \mathbb{A}_{c_i}(-\tfrac{1}{2}), \quad \dim \mathbb{A}_{c_i}(-\tfrac{1}{2}) = 2$$
2. $\mathbb{E}_3 \cong \mathrm{Tri}(\mathbf{H}_1) = \mathrm{Tri}(\mathbb{K})$.
3. The corresponding eigencubic is $u_{\mathbb{E}_3} = x_1x_2x_3$, and the minimal cone is the triple of coordinate planes in $\mathbb{R}^3$.
4. $\mathrm{Aut}(\mathbb{E}_3) = S_4$ (permuting the four idempotents).

Remark. $\mathbb{E}_3$ is one of two Hsiang algebras having finite number of idempotents and therefore a finite automorphism group!

Basic facts on Hsiang algebras

The set of nonzero idempotents in any Hsiang algebra $\mathbb{A}$ is nonempty and all idempotents **have the same length and the same fusion laws**. In particular, for any idempotent c, the associated **Peirce decomposition** is

$$\mathbb{A} = \underbrace{\mathbb{A}_c(1)}_{\dim=1} \oplus \underbrace{\mathbb{A}_c(-1)}_{\dim=n_1} \oplus \underbrace{\mathbb{A}_c(-\tfrac{1}{2})}_{\dim=n_2} \oplus \underbrace{\mathbb{A}_c(\tfrac{1}{2})}_{\dim=n_3}$$

- $\mathbb{A}_c(1) \oplus \mathbb{A}_c(-1)$ is a subalgebra. It carries a **hidden Clifford algebra structure**, $n_1 = \dim \mathbb{A}_c(-1)$
- $\mathbb{A}_c(1) \oplus \mathbb{A}_c(-\frac{1}{2})$ is a subalgebra. It carries a **hidden rank 3 Jordan algebra structure**, $n_2 = \dim \mathbb{A}_c(-\frac{1}{2})$.
- $\mathbb{A}$ is **exceptional if and only if** $\mathbb{A}_c(1) \oplus \mathbb{A}_c(-\frac{1}{2})$ is (isotopy of) a simple Jordan algebra. In this case, either $n_2 = 0$ or $n_2 = 3\mathbf{d} + 2$ and the hidden simple Jordan algebra is $\mathrm{Herm}_3(\mathbf{H_d})$, $\mathbf{d} \in \{1, 2, 4, 8\}$.
- $\mathbb{A}$ is **mutant iff** $n_2 = 2$, this corresponds to $\mathbf{d} = 0$.
- $\mathbb{A}$ is **exceptional or mutant iff** $\operatorname{tr} L(x)^2 = m\langle x; x\rangle$ for some real m. In this case, $m = 2(n_1 + \mathbf{d} + 1)$.
- There are **finitely many dimensions** n of $\mathbb{A}$ where exceptional Hsiang algebras can exist. Except the case $n_2 = 0$, in all other cases, $\dim \mathbb{A} = \mathbf{3}(n_1 + 2\mathbf{d} + 1)$, where $\dim \mathbb{A}_c(-\frac{1}{2}) = 3\mathbf{d} + 2$, $\mathbf{d} \in \{0, 1, 2, 4, 8\}$.

n	2	5	8	14	26	3	6	12	24	9	12	21	15	18	30	42	27	30	54
n_1	1	2	3	5	9	0	1	3	7	0	1	4	0	1	5	9	0	1	1
n_2	0	0	0	0	0	2	2	2	2	5	5	5	8	8	8	8	14	14	26
$\mathbf{d}$	–	–	–	–	–	0	0	0	0	1	1	1	2	2	2	2	4	4	8

The colours correspond to pseudo-composition, mutants and two unsettled cases.

An important tool to study a bilinear form is to diagonalize it. For cubic forms the situation is more subtle. In the context of metrized algebras, there are at least two distinguished ways to write $u(x)$ in some orthonormal coordinates in $\mathbb{R}^n$:

1. **Normal form**:

$$u(x) = x_1^3 + 3\underbrace{\left(-1 \cdot |\xi|^2 - \tfrac{1}{2} \cdot |\eta|^2 + \tfrac{1}{2} \cdot |\zeta|^2\right)x_1}_{\text{the Peirce decomposition}} + \underbrace{\psi(\xi, \eta, \zeta)}_{\text{fusion laws}}.$$

   Here $x = (0, 1)$ corresponds to an idempotent in $\mathbb{A}(u)$, see for example (2).

2. **Steiner form**:

$$u(x) = \sum_{\alpha \in B} \epsilon_\alpha \underbrace{(x_{\alpha_1} x_{\alpha_2} x_{\alpha_3})}_{\text{the skeleton idempotents}}, \qquad \epsilon_\alpha \in \mathbb{K}.$$

   where B a **partial Steiner triple system** (*PSTS*) on $\mathbb{N}_n$. Not every cubic form admits a Steiner form. But, for a Hsiang eigencubic, if a Steiner form exists then the coefficients ϵ_α must be ± 1.

Example: the determinant eigencubic in $9D$

The following are Hsiang eigencubics in $\mathbb{K}^{6d+3}$ (Hsiang,1967; Hoppe, V.T., 2018):

$$u_d(X) := \operatorname{tr} X^3, \qquad X \in \mathrm{Herm}(\mathbf{H}_d, 4), \quad \operatorname{tr} X = 0, \quad d \in \{1, 2, 4\}$$

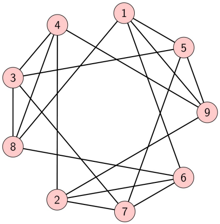
For $d = 1$, this provides an **exceptional** eigencubic $u_1(x)$ in $\mathbb{K}^9$ which in some orthonormal coordinates can be written as the determinant:

$$u_1(x) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{vmatrix} = \underbrace{\begin{matrix} x_1x_5x_9 + x_2x_6x_7 + x_3x_4x_8 \\ -x_1x_6x_8 - x_2x_4x_9 - x_3x_5x_7 \end{matrix}}_{\text{a Steiner form}},$$

where the set of **unordered** triples

$$B = \{(1,5,9), (2,6,7), (3,4,8), (1,6,8), (2,4,9), (3,5,7)\}$$

is a **regular** partial Steiner triple system on $\mathbb{N}_9$ with **replication number** $r = 2$. One can naturally assign to B a 4-regular graph (two vertices i, j are incident iff $\{i, j, \star\} \in B$):



A handicap graph (Kovar, Kravčenko) et al, 2017

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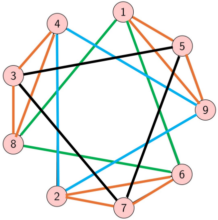


A handicap graph (Kovar, Kravčenko) et al, 2017

Let $\mathbb{N}_n := \{1, \ldots, n\}$. A collection B of triples from $\mathbb{N}_n$, so that each element $i \in \mathbb{N}_n$ occurs in at least one triple and each unordered pair $i \neq j$ occurs in at most one triple of B, is called a *PSTS*.

Given a *PSTS* on $\mathbb{N}_n$, there is smallest subset $R \subset \mathbb{N}$ such that each $i \in \mathbb{N}_n$ is contained in exactly $r \in R$ blocks; R is called the **set of replication numbers**. If $R = \{r\}$ then the *PSTS* is called **regular**.

Example

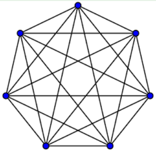
A 2-regular *PSTS* on $\mathbb{N}_6$:

$$\{(1,3,6),(1,4,5),(2,3,5),(2,4,6)\},$$

A $\{2,4\}$-**regular** on $\mathbb{N}_{10}$:

$$\{(\mathbf{1},3,5),(\mathbf{1},4,6)(\mathbf{1},7,9),(\mathbf{1},8,10),(\mathbf{2},3,6),(\mathbf{2},4,5),(\mathbf{2},7,10),(\mathbf{2},9,8)\}, \quad (4)$$

The Fano plane on $\mathbb{N}_7$,
$\{(1,2,3),(1,4,5),(1,6,7),(2,4,6),(2,5,7),(3,4,7),(3,5,6)\}$,
is **regular**, $r = 3$ and, in fact, it is a Steiner triple (all pairs occur), such a triple exists iff $n = 6k+1$ or $6k+3$.



Steiner form

A cubic form $u(x)$ on an inner product vector space $(V, \langle;\rangle)$, $\dim V = n$, is said to admit a **Steiner form** if there exist a *PSTS* B on $\mathbb{N}_n$ and a basis e_i such that

$$u(x) = \sum_{\alpha \in B} a_\alpha x_{\alpha_1} x_{\alpha_2} x_{\alpha_3}, \qquad x_i = h(x, e_i), \quad a_\alpha \in \mathbb{K}.$$

An orthonormal basis $\{e_i\}$ of a metrized algebra $(\mathbb{A}, \langle;\rangle)$ is said to be a **Steiner basis** if $e_i e_i = 0$ for all i and $e_i e_j$ is proportional to some e_k for all $i \neq j$.

Proposition

Let $(\mathbb{A}, \langle;\rangle)$ be a metrized algebra, $(\mathbb{A}\mathbb{A})^{\perp} = 0$. The defining form $u_{\mathbb{A}}(x) = \frac{1}{6}\langle x^2; x\rangle$ admits a Steiner form if and only $\mathbb{A}$ admits a Steiner basis.

Proof follows from $u_{\mathbb{A}}(x) = \sum_{i,j,k} \langle e_i e_j; e_k\rangle x_i x_j x_k$. To see that this is indeed a Steiner form, suppose that there two terms $x_i x_j x_k$ and $x_i x_j x_m$ appear in the latter decomposition. Then both $\langle e_i e_j; e_k\rangle$ and $\langle e_i e_j; e_m\rangle$ are nonzero, a contradiction. Since $(\mathbb{A}\mathbb{A})^{\perp} = 0$, then for any i: $\langle e_i \mathbb{A}; \mathbb{A}\rangle = \langle e_i; \mathbb{A}\mathbb{A}\rangle \neq 0$, hence there exists j such that $e_i e_j \neq 0$. □

Proposition (Skeleton idempotents)

Let $\{e_i\}$ be a Steiner basis of $\mathbb{A}$.

1. If $e_ie_j = \lambda e_k$, $\lambda \neq 0$, then $e_je_k = \lambda e_k$ and $e_ke_i = \lambda e_j$.
2. If $\alpha^2 = \beta^2 = \gamma^2 = \alpha\beta\gamma = 1$ then

$$c = c_{\alpha,\beta,\gamma} := (\alpha e_i + \beta e_j + \gamma e_k)/2\lambda$$

   is a idempotent (called a **skeleton** idempotent) and $\langle c; c\rangle = 3\lambda^2/4$.
3. The skeleton idempotents span $\mathbb{A}$ as a vector space.
4. $\mathrm{span}(e_i, e_j, e_k)$ is a subalgebra of $\mathbb{A}$ isomorphic to $\mathbb{E}_3$

Proof. If $e_ie_j \neq 0$ then $e_ie_j = \lambda e_k$, $\lambda \in \mathbb{K}^{\times}$, hence

$$\langle e_ie_j; e_k\rangle = \langle e_je_k; e_i\rangle = \langle e_ke_i; e_j\rangle = \lambda$$

implies that $e_je_k = \lambda e_k$ and $e_ke_i = \lambda e_j$. This yields

$$(\alpha e_i + \beta e_j + \gamma e_k)^2 = 2\lambda(\beta\gamma e_i + \alpha\gamma e_j + \alpha\beta e_k) = 2\lambda(\alpha e_i + \beta e_j + \gamma e_k).$$

The **four** distinct idempotents $c_{\alpha,\beta,\gamma}$ satisfy the axioms of $\mathbb{E}_3$.

Corollary 1

Let $\mathbb{A}$ be a Hsiang algebra satisfying (1), $\{e_i\}$ be a Steiner basis of $\mathbb{A}$ and B the corresponding *PSTS*. Then for $a = \sqrt{3/4\theta}$

$$e_i e_j = \begin{cases} 0, & (i,j,k) \notin B; \\ \pm a e_k, & (i,j,k) \in B. \end{cases} \tag{5}$$

and in some orthonormal coordinates $u_{\mathbb{A}} = \sum_{\alpha \in B} \pm(x_{\alpha_1} x_{\alpha_2} x_{\alpha_3})$

Corollary 2

For any $s \in \mathbb{N}_n$, $n = \dim \mathbb{A}$,

$$L(e_s)^2 : \mathbb{A} \to W_s$$

is the orthogonal projection onto $W_s = \operatorname{span}(\{e_j : e_j e_s \neq 0\})$. Moreover,

$$\dim W_s = 2r, \qquad r \in R \text{ is the replication number of } s.$$

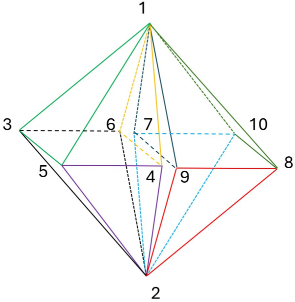
$\mathbb{E}_3$-complex: each triple

$$x_{\alpha_1} x_{\alpha_2} x_{\alpha_3}$$

corresponds to a simplicail cell

$$\mathbb{E}_3 = \operatorname{span}(e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_3})$$

See (4) above for $n = 10 \Longrightarrow$



$n = 10$
$R = \{2, 4\}$
$\mathbb{A} = \mathbb{A}_0 \oplus \mathbb{A}_1$

Theorem

Let a Hsiang algebra $\mathbb{A}$ admit a Steiner form B on $\mathbb{N}_n$. Then $n_2 \geq 2$ and either of the following holds:

1. $\mathbb{A}$ is **exceptional or mutant**, B is r-regular, where

$$r = n_1 + d + 1, \qquad |B| = (n_1 + d + 1)(n_1 + 2d + 1).$$

2. $\mathbb{A}$ is a **polar algebra** $\mathbb{A}_0 \oplus \mathbb{A}_1$, B is R-regular, where

$$R = \{\dim \mathbb{A}_0, \tfrac{1}{2} \dim \mathbb{A}_1\}, \qquad |B| = \tfrac{1}{2} \dim \mathbb{A}_1 \dim \mathbb{A}_0.$$

Hsiang defining forms in 15 and 18 dimensions

A Steiner type form: $n = 15$, $|B| = 15$, replication number $r = 3$

$$x_1x_6x_{15} - x_1x_9x_{14} + x_1x_{10}x_{13} - x_2x_5x_{15} + x_2x_8x_{14} - x_2x_{10}x_{12} + x_3x_4x_{15} - x_3x_7x_{14} + x_3x_{10}x_{11} - x_4x_8x_{13} + x_4x_9x_{12} + x_5x_7x_{13} - x_5x_9x_{11} - x_6x_7x_{12} + x_6x_8x_{11}$$

A Steiner type form: for $n = 18$, $|B| = 24$, replication number $r = 4$

$$x_1x_5x_9 - x_1x_6x_8 - x_1x_{14}x_{18} + x_1x_{15}x_{17} - x_4x_2x_9 + x_7x_2x_6 + x_{13}x_2x_{18} - x_{16}x_2x_{15} + x_4x_3x_8 - x_7x_3x_5 - x_{13}x_3x_{17} + x_{16}x_3x_{14} + x_4x_{11}x_{18} - x_4x_{12}x_{17} - x_{10}x_5x_{18} + x_{16}x_{12}x_5 + x_{10}x_6x_{17} - x_{16}x_{11}x_6 - x_7x_{11}x_{15} + x_7x_{12}x_{14} + x_{10}x_{15}x_8 - x_{13}x_{12}x_8 - x_{10}x_{14}x_9 + x_{13}x_{11}x_9$$

For exceptional and mutant Hsiang cubics in general always holds true:

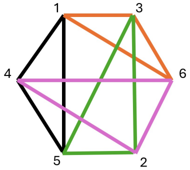
$$|B| = \frac{n \cdot r}{3}$$

Example H_6

$$H_6 = \{(1,3,6),(1,4,5),(2,3,5),(2,4,6)\}$$

$$u = x_1x_3x_6 - x_1x_4x_5 + x_2x_3x_5 + x_2x_4x_6$$

$$v = x_1x_3x_6 + x_1x_4x_5 + x_2x_3x_5 + x_2x_4x_6$$

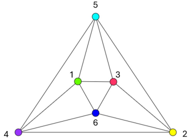


The corresponding algebras are

$$\mathbb{A}(u) \cong \mathrm{Tri}(\widehat{\mathbb{C}}) \cong \mathbb{E}_2 \otimes \mathbb{E}_3$$

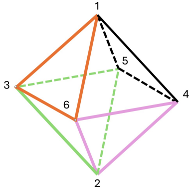
$$\mathbb{A}(v) \cong \mathrm{Tri}(\mathrm{Cl}(1,0)) \cong \mathbb{E}_3 \otimes \mathrm{Cl}(1,0)$$

($\mathrm{Cl}(1,0)$ is the algebra of split-complex numbers)



The corresponding graph is the octahedron:

(*Note that not all trianglular faces are triples*!)



The sign decoration does matter!

The cubic forms

$$\operatorname{perm}(A) = x_1x_5x_9 + x_2x_6x_7 + x_3x_4x_8 + x_1x_6x_8 + x_2x_4x_9 + x_3x_5x_7,$$
$$\det(A) = x_1x_5x_9 + x_2x_6x_7 + x_3x_4x_8 - x_1x_6x_8 - x_2x_4x_9 - x_3x_5x_7,$$

have the same KIlling invariant form $\operatorname{tr} L(x)L(y)$ but they give rise to non-isomorphic algebras! More precisely,

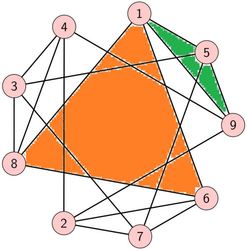
Theorem (D. Fox 2020, D. Fox, V.T, 2025)

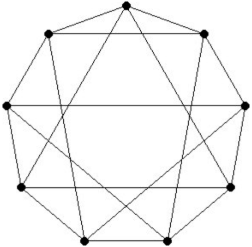
$$\mathrm{Det} \cong \mathfrak{so}(3, \mathbb{K}) \otimes \mathfrak{so}(3, \mathbb{K})$$
$$\mathrm{Perm} \cong \mathbb{E}_3 \otimes \mathbb{E}_3,$$

where $\mathbb{E}_3$ is the simplicial algebra. Moreover, Det and $\mathbb{E}_3$ are Hsiang algebras, but not Perm.

Existence of a triple decomposition

To any 'Hsiang' *PSTS* one can naturally associate a $2r$-regular planar indirected graph, but the converse is not true, because such a graph must additionally be a disjoint union of triangles (corresponding to triples in B):





Another necessary condition for B to be Hsiang is

$$|B| = n \cdot r/3. \tag{6}$$

But not any *PSTS* satisfying (6) is Hsiang. For example, since for any idempotent c in a Hsiang algebra, $L(c)$ does not contain a zero eigenvalue, it follows from (5) that any $\mathbb{E}_3$-face and any vertex of the associated graph are incendent.

D. Fox, V.G. Tkachev, *Algebraic constructions of cubic minimal cones*, Pure Appl. Funct. Anal., 2025 (to appear)

Thank you!