Ore extensions of abelian groups with operators

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Talk based on joint work

- P. Bäck, P. Lundström, J. Öinert and J. Richter. *Ore extensions of abelian groups with operators*
- Available at https://arxiv.org/abs/2410.16761

Motivating question

When are "polynomesque" structures noetherian?

Rings and modules

- Let R be a *ring*. By this we mean that R is an additive group with a map $R \times R \ni (r, s) \mapsto rs \in R$.
- By a left R-*module* we mean an additive group M with a map $R \times M \ni (r, m) \mapsto rm \in M$.
- Every ring is a left module over itself.
- A left R-module M is said to be:
  - *Left distributive* if $r(m + n) = rm + rn$, $r \in R$, $m, n \in M$.
  - *Right distributive* if $(r + s)m = rm + sm, r, s \in R, m \in M$.
  - *Associative* if $(rs)m = r(sm)$, $r, s \in R$, $m \in M$.

Noetherian rings and modules

- Suppose M is a left R-module.
- An additive subgroup N of M is called an *R-submodule* if $RN \subseteq N$.
- M is said to be *Noetherian* if any chain $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_i \subseteq \cdots$ of R-submodules of M eventually stabilizes, that is if there is some $k \in \mathbb{N}$ such that $N_i = N_k$ for every $i \geq k$.
- R is said to be left Noetherian if it is Noetherian as a left module over itself. Then submodule $\Leftrightarrow$ ideal.

For now. . .

All rings and modules are assumed to be both left and right distributive.

Noetherian background

Hilbert's basis theorem (Hilbert 1890)

Suppose R is an associative and unital ring. Then $R[X]$ is left/right noetherian $\Leftrightarrow$ R is left/right noetherian.

A skew Hilbert's basis theorem (Noether and Schmeidler 1920)

Suppose R is an associative and unital ring and let σ be a ring automorphism of R. If R is left/right noetherian, then the skew polynomial ring $R[X;\sigma]$ is left/right noetherian.

Noetherian background

An Ore extension Hilbert's basis theorem (Cohn 1971; Faith 1973)

Suppose R is an associative and unital ring. Let σ be a ring automorphism of R and let δ be a σ-derivation of R. If R is left/right noetherian, then the Ore extension $R[X;\sigma,\delta]$ is left/right noetherian.

Noetherian background

A hom-associative Hilbert's basis theorem (Bäck and Richter 2018)

Let R be a hom-associative and unital ring with twisting map α. Let σ be a ring automorphism of R and let δ be a σ-derivation that both commute with α. Extend α homogeneously to $R[X;\sigma,\delta]$. If R is left/right noetherian, then the nonassociative Ore extension $R[X;\sigma,\delta]$ is left/right noetherian.

Noetherian background

A polynomial module Hilbert's basis theorem (Varadarajan 1982)

Suppose that R is an associative, but not necessarily unital, ring and M is an associative left R-module. Then the left $R[x]$-module $M[x]$ is Noetherian if and only if M is Noetherian and s-unital (that is $m \in Rm$ for $m \in M$).

Natural questions

- Is it possible to define a class of "Ore module extensions" so that these simultaneously generalize polynomial modules and classical Ore extensions?
- If so, can algebraical structure results for Ore module extensions, such as associativity and a Hilbert's basis theorem, be established?

MAIN RESULTS

B is an abelian group with operators in a nonempty set A.

- *Let* $B[x;\sigma_B,\delta_B]$ *be an Ore group extension of a stably Noetherian abelian group* B *on which the action of* A *is weakly* **s***-unital. Let* σ_B *be an* A*-stable surjection of* B. *Then* $B[x;\sigma_B,\delta_B]$ *is stably Noetherian, seen as a group with operators in* $A[x]$.
- *Consider* $B[x]$ *as an abelian group with operators in* $A[x]$. *Then* $B[x]$ *is stably Noetherian* $\Leftrightarrow$ B *is stably Noetherian and the action on* B *is weakly* **s***-unital.*

Krull (1925) and Noether (1929)

- Let $\alpha : A \to B^B$ be an *action* of a set A on a set B.
- By abuse of notation $ab := \alpha(a)(b)$, $a \in A$, $b \in B$.
- If $(B, \cdot)$ is a group, then it is called a *group with operators in* A if $a(b \cdot c) = (ab) \cdot (ac), a \in A, b, c \in B$.
- Note that no assumption is made on associativity of the action since $a_1(a_2 b) = (a_1 a_2)b$ makes no sense!

Folklore (Jacobson, Bourbaki, ...)

- Suppose B is a group with operators in A. Let $S \subseteq B$.
- S is called *stable* if $AS \subseteq S$.
- The intersection of the family of stable subgroups of B that contain S is called the stable subgroup of B *generated* by S and is denoted by $\langle S \rangle$.
- If C is a stable subgroup of B such that $C = \langle T \rangle$ for some finite subset T of B, then C is said to be *finitely generated* by T.

Folklore

Consider the partially ordered set of stable subgroups of B, ordered by inclusion. We say that B is *stably Noetherian* if this partially ordered set satisfies the ascending chain condition.

Proposition

B is stably Noetherian $\Leftrightarrow$ any nonempty family of stable subgroups of B has a maximal element $\Leftrightarrow$ every stable subgroup of B is finitely generated.

Stable homomorphisms

Let B and C be groups with operators in A. Suppose that $f : B \to C$ is a group homomorphism. Then we say that f is *A-stable* if for every $b \in B$, $f(Ab) \subseteq Af(b)$ holds.

Proposition

Let B be a stably Noetherian group with operators in A. Suppose that $f : B \to B$ is a surjective A-stable group endomorphism. Then f is bijective.

Twisted homomorphisms

Let B and C be groups with operators in A. Suppose that $f : B \to C$ is a group homomorphism. Let τ be a map $A \to A$. We say that f is τ*-twisted* if for all $a \in A$ and all $b \in B$, the equality $f(ab) = \tau(a)f(b)$ holds.

Proposition

Let B and C be groups with operators in A. Suppose that $f : B \to C$ is a τ-twisted group homomorphism for some map $\tau : A \to A$. Then f is A-stable.

Weakly s-unital action

Suppose B is a group with operators in A and let $S \subseteq B$. Put $\widetilde{S} := \cup_{n \in \mathbb{N}_+}(A^n S)$. Let $[S]$ denote the set of all $b_1 \cdots b_n$, for $n \in \mathbb{N}_+$, where for each $k \in \{1, \ldots, n\}$, $b_k \in \widetilde{S}$ or $b_k \in \left(\widetilde{S}\right)^{-1}$. We say that the action of A on B is *s-unital* (resp. *weakly s-unital*) if for every $b \in B$ the relation $b \in Ab$ (resp. $b \in [b]$) holds.

Example

$A = \{a\}$, $B = C_3$, $ab = b^{-1} \Rightarrow$ weakly s-unital action which is not s-unital.

Abelian groups with operators

- Suppose that $(B, +, 0)$ is an *abelian* group with operators in a nonempty set A.
- We always assume that A has a *zero element*. By this, we mean an element $\epsilon \in A$ such that for any $b \in B$, $\epsilon b = 0$. We will assume that ϵ is fixed.
- By abuse of notation, we put $0 := \epsilon$, so that $a0 = 0b = 0$ holds for all $a \in A$ and $b \in B$.

Polynomial groups

By a *polynomial* over A we mean a formal sum $\sum_{i \in \mathbb{N}} a_i x^i$, where $a_i \in A$, for $i \in \mathbb{N}$, and $a_i = 0$ for all but finitely many $i \in \mathbb{N}$. The set of polynomials over A is denoted by $A[x]$. We define $B[x]$ similarly and equip it with an abelian group structure in the following way. If $\sum_{i \in \mathbb{N}} b_i x^i, \sum_{i \in \mathbb{N}} b'_i x^i \in B[x]$, then we put

$$\sum_{i \in \mathbb{N}} b_i x^i + \sum_{i \in \mathbb{N}} b'_i x^i := \sum_{i \in \mathbb{N}} (b_i + b'_i) x^i.$$

The zero polynomial is defined to be $0 := \sum_{i \in \mathbb{N}} 0 x^i$.

π-maps

Let σ_B and δ_B be group endomorphisms of B. Take $i, j \in \mathbb{N}$. We define $\pi^i_j \colon B \to B$ in the following way.

- If $i \geq j$, then we let $\pi^i_j \colon B \to B$ denote the sum of all $\binom{i}{j}$ compositions of j instances of σ_B and $i - j$ instances of δ_B.
- If $i < j$, then we put $\pi^i_j := 0$.

Ore group extension

The *Ore group extension* $B[x;\sigma_B,\delta_B]$ is the abelian group $B[x]$ having $A[x]$ as a set of operators, with action:

$$\left(\sum_{i\in\mathbb{N}} a_i x^i\right)\left(\sum_{j\in\mathbb{N}} b_j x^j\right) := \sum_{i,j,k\in\mathbb{N}} \left(a_i \pi_k^i(b_j)\right) x^{k+j}$$

for $\sum_{i\in\mathbb{N}} a_i x^i \in A[x]$ and $\sum_{j\in\mathbb{N}} b_j x^j \in B[x]$.
By abuse of notation, we write $B[x]$ for $B[x;\mathrm{id}_B,0_B]$.

σ-derivation and σ-twist

Suppose from now on that σ_A and δ_A are maps $A \to A$ and that σ_B and δ_B are additive maps $B \to B$.

- We say that δ_B is a *δ_A-twisted σ_A-derivation* if $\delta_B(ab) = \sigma_A(a)\delta_B(b) + \delta_A(a)b$ for $a \in A$ and $b \in B$.
- We say that σ_B is *σ_A-twisted* if $\sigma_B(ab) = \sigma_A(a)\sigma_B(b)$ for all $a \in A$ and $b \in B$.

Classical identities

Vandermonde's identity

$$\sum_{i\in\mathbb{N}} \pi_i^k \circ \pi_{j-i}^n = \pi_j^{k+n}$$

Leibniz's identity

$$\pi_i^m(ab) = \sum_{k\in\mathbb{N}} \pi_k^m(a)\pi_i^k(b)$$

Mixed Vandermonde's and Leibniz's identity

$$\sum_{i\in\mathbb{N}} \pi_i^m(a\pi_{j-i}^n(b)) = \sum_{i\in\mathbb{N}} \pi_i^m(a)\pi_j^{i+n}(b)$$

MAIN RESULTS

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- *Consider $B[x]$ as an abelian group with operators in $A[x]$. Then $B[x]$ is stably Noetherian $\Leftrightarrow$ B is stably Noetherian and the action on B is weakly* **s**-*unital.*

Applications

The Cayley-Dickson doubling procedure

$$\overline{(a,b)} := (\bar{a}, -b) \quad \text{and} \quad (a,b)(c,d) := (ac - d\bar{b}, cb + \bar{a}d)$$

- $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H} \subseteq \mathbb{O} \subseteq \mathbb{S} \subseteq \cdots$ left/right distributive
- Finite-dimensional $\Rightarrow$ Noetherian.
- $\mathbb{C}$ not ordered.
- $\mathbb{H}$ not commutative.
- $\mathbb{O}$ not associative.
- $\mathbb{S}$ has zero divisors (hence norm not multiplicative).

Applications

The Conway-Smith doubling procedure

$$(a,b)(c,d) := \begin{cases} \left(ac - \overline{b\overline{d}}, \overline{\overline{b}\,\overline{c}} + \overline{\overline{b}\,\overline{\overline{a}\,\overline{\overline{b^{-1}}\,\overline{d}}}}\right) & \text{if } b \neq 0 \\ (ac, \overline{a}d), & \text{otherwise} \end{cases}$$

- $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H} \subseteq \mathbb{O} \subseteq$ 16-ons $\subseteq$ 32-ons $\subseteq \cdots$
- Finite-dimensional $\Rightarrow$ Noetherian.
- Left distributive.
- The 16-ons are not right distributive.
- All of the 2^n-ons are norm multiplicative!

Applications

- Dickson's *left near-fields* (1906) are *finite* rings, hence in particular Noetherian. Left near-fields are left distributive but not always right distributive.
- $\mathbb{F}_2 = \{0, 1\}$ field with two elements. $G = \{R, P, S\}$ is the *rock, paper, scissors* magma: $R^2 = R$, $P^2 = P$, $S^2 = S$, $RP = P$, $RS = R$, $PS = S$. The magma algebra $\mathbb{F}_2[G]$ is a Boolean ring. In particular, $\mathbb{F}_2[G]$ is weakly left s-unital. Since $\mathbb{F}_2[G]$ is finite, it is Noetherian. Therefore, $\mathbb{F}_2[G][x]$ is left Noetherian.

Thank you!