Generalized Weyl Algebras and their representations

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Based on current work with Samuel Lopes (Universidade do Porto).

Generalized Weyl Algebras

Definition

Let R be a ring, $a \in Z(R)$ a central element, and $\sigma \in \mathrm{Aut}(R)$ a ring automorphism. The corresponding **Generalized Weyl Algebra** $R(\sigma, a)$ is generated by R and two variables x and y with relations

$$xr = \sigma^{-1}(r)x \qquad yr = \sigma(r)y \qquad xy = a \qquad yx = \sigma(a)$$

Realizing various algebras as GWAs

Algebra $R(\sigma, a)$	**Ring** R	**Automorphism** σ	**Element** a
Classical Weyl Algebra	$k[h]$	$\sigma(h) = h - 1$	h
Quantum torus	$k[t, t^{-1}]$	$\sigma(t) = qt$	t
Quantum Weyl Algebra	$k[h]$	$\sigma(h) = qh - 1$	h
$U(\mathfrak{sl}_2)$/central action	$\mathbb{C}[h]$	$\sigma(h) = h - 2$	$\frac{1}{4}h(h+2)$
Smith algebras/central action	$k[h]$	$\sigma(h) = h - 1$	$p(h) \in k[h]$

Weight modules for GWAs

Assume from now on that R is a UFD. Let $A = R(\sigma, a)$ be a GWA and let V be an A-module. For each maximal ideal $\mathfrak{m} \subset R$ we define the corresponding **weight space**:

$$V_{\mathfrak{m}} = \{v \in V \mid \mathfrak{m}v = 0\}.$$

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V is called a **weight module** if $V = \bigoplus_{\mathfrak{m} \in \mathrm{supp}(V)} V_{\mathfrak{m}}$.

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Definition

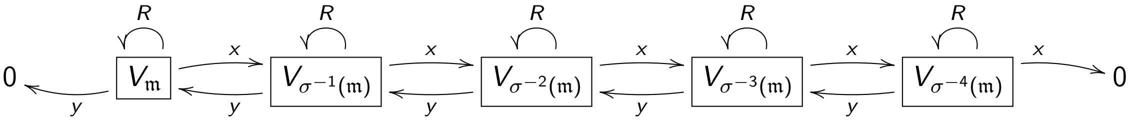
For $\mathfrak{m} \in \mathrm{Max}(R)$ Let $R_{\mathfrak{m}}$ be the $R[y]$-module which is equal to $R/\mathfrak{m}$ as an R-module, and where y acts as zero. Define the corresponding Verma module as the A-module

$$V(\mathfrak{m}) = R(\sigma, a) \otimes_{R[y]} R_{\mathfrak{m}}.$$

Then $V(\mathfrak{m})$ has a unique maximal submodule and a corresponding simple quotient $L(\mathfrak{m})$.

Visualization

Visualization of a simple weight module $L(\mathfrak{m})$ with finite support:



Lifting free modules for Lie algebras - previous work

For a Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, consider the following subcategory of $U(\mathfrak{g})$-Mod:

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- Classification $\mathfrak{C}_1$ modules for $\mathfrak{g} = \mathfrak{sl}_n$ (N. 2015)
- Classification of $\mathfrak{C}_1$ modules for $\mathfrak{g} = \mathfrak{sp}_{2n}$ (N. 2016)
- Cartan-free modules do not exist for any other type of Lie algebras (N. 2016)
- Simple $\mathfrak{sl}_2$-modules in $\mathfrak{C}_n$ for arbitrary rank n (F. Martin, C. Prieto 2017)
- Generalizations and extensions of $\mathfrak{C}_n$ for other types of algebras:
    - Virasoro algebras (G. Liu, K. Zhao)
    - Conformal algebras (Q. Xie et al.)
    - The Witt algebra (H. Tan, K. Zhao)
    - Algebras of differential operators (S. Gao et al.)
    - Heisenberg-Virasoro algebras (H. Chen, X. Guo)
    - Super Lie algebras (Y. Cai, K. Zhao)
    - Kac-Moody algebras (K. Zhao et al.)
    - Smith algebras (V. Futorny, S. Lopes, E. Mendonca)

Two category of modules for GWAs

Definition

For a GWA $R(\sigma, a)$, consider the full subcategories of $R(\sigma, a)$-Mod:

$$\mathfrak{C} = \{M \in R(\sigma, a)\text{-Mod} \mid M \text{ is finitely generated over } R\}.$$

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This is an abelian category. We define also $\mathfrak{C}_n \subset \mathfrak{C}$:

$$\mathfrak{C}_n = \{M \in R(\sigma, a)\text{-Mod} \mid M \simeq_R R^n\},$$

Objects of $\mathfrak{C}_1$

Definition

Let $R(\sigma, a)$ be a GWA. Let $p|a$ be a divisor, and define $q = \sigma(a/p)$. We define a corresponding $R(\sigma, a)$-module V_p, which as a set (and R-module) is R and where the action is given by:

$$x \cdot r = \sigma^{-1}(r)p \quad \text{and} \quad y \cdot r = \sigma(r)q.$$

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Then any module in $\mathfrak{C}_1$ is isomorphic to some V_p, and $V_p \not\simeq V_{p'}$ unless p and p' are associates.

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Let W be a submodule of the $R(\sigma, a)$-module V_p. Since R is a PID we have $W = \langle g \rangle$, and the submodules form a lattice. The GWA relations force $g|\sigma^{-1}(g)p$ and $g|\sigma(g)q$.

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So if $g = \prod g_i$ is a complete factorization, we get

$$g_i|\sigma^{-1}(g) \text{ or } g_i|p$$

and

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σ-orbits

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$$r = \prod_{\omega \in \Omega} r_\omega \quad \text{where all irreducible factors of } r_\omega \text{ lies in } \omega.$$

Then $\langle g \rangle \subset V_p$ is a $R(\sigma, a)$-submodule if and only if $\langle g_\omega \rangle \subset V_{p_\omega}$ is a $R(\sigma, a_\omega)$-submodule for each $\omega \in \Omega$.

Submodules - infinite orbit case

Assume that all factors of a lie in a single *infinite* σ-orbit.
Then the maximal submodules of V_p have form $W = \langle g \rangle$ where g is a chain-product

$$g = \prod_{i=0}^{n} \sigma^i(z)$$

where $z | q$ and $\sigma^n(z) | p$, and such that the factors in the middle of the chain divides neither p nor q.

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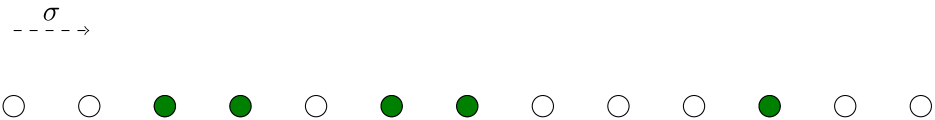
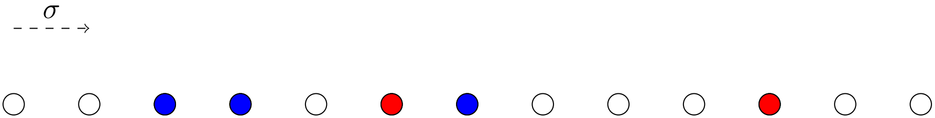
We note that $W \simeq V_{p'}$ where $p' = \frac{p}{\sigma^n(z)}\sigma^{-1}(z)$.



factors of a



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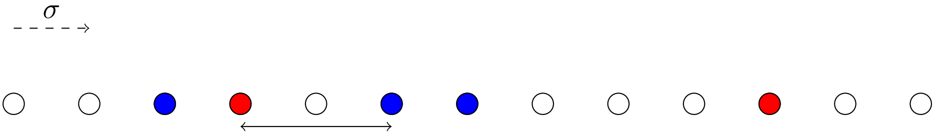

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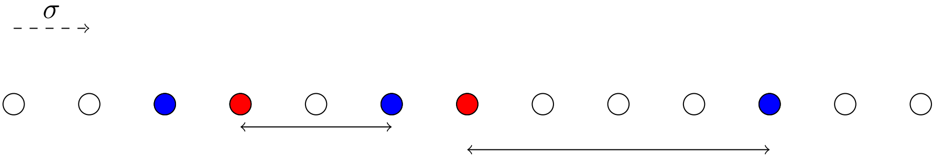

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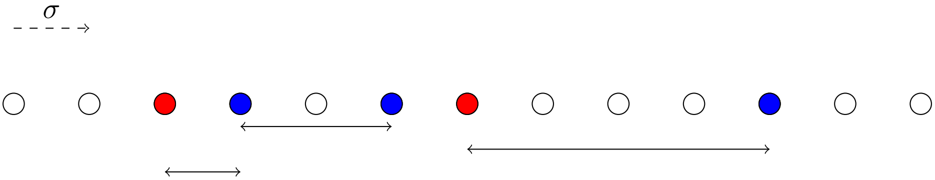

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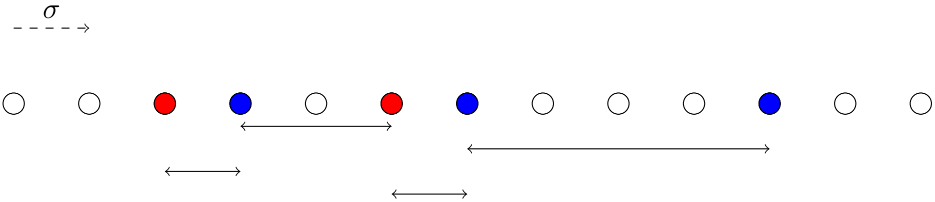

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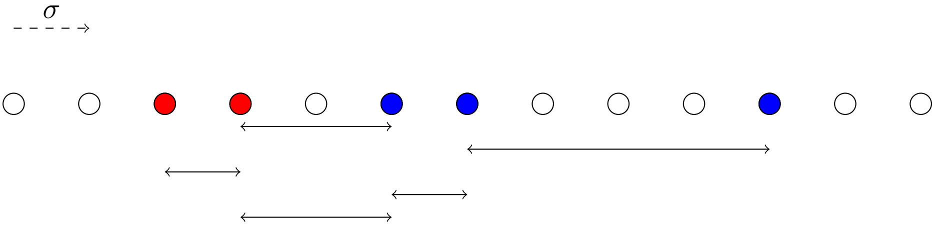

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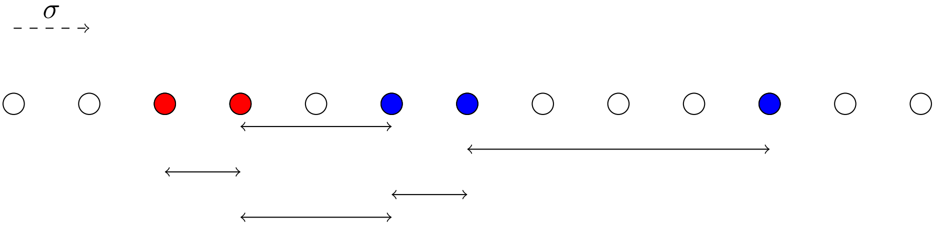

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Quotients

Assume that σ-orbits are infinite. Let $\langle g \rangle$ be a maximal submodule of V_p, with $g = \prod_{i=0}^{n} \sigma^i(z)$ and $z|q$ and $\sigma^n(z)|p$.
As R-modules we have

$$V_p/\langle g \rangle \simeq R/\langle z \rangle \oplus R/\langle \sigma(z) \rangle \oplus \cdots \oplus R/\langle \sigma^n(z) \rangle,$$

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As R-modules we have

$$V_p/\langle g \rangle \simeq R/\langle z \rangle \oplus R/\langle \sigma(z) \rangle \oplus \cdots \oplus R/\langle \sigma^n(z) \rangle,$$

This is in fact isomorphic to the simple *weight module* $L(\mathfrak{m})$ for $\mathfrak{m} = \langle \sigma^n(z) \rangle$.

Length in infinite orbits case

The length of the module V_p is the *number of flips* in our diagram. When a is square free this is equal to the number of pairs (p_i, q_i) of irreducible factors of p and q respectively where $p_i \in \sigma^{\mathbb{N}}(q_i)$.

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We get a bound

$$\operatorname{len}(V_p) \leq \left(\frac{\deg(a)}{2}\right)^2 + 1$$

and $\mathfrak{C}_1$ a finite length category.

Grothendieck group

Definition

Recall that Grothendieck group $K_0(\mathfrak{C})$ is the abelian group generated by iso-classes of modules in $\mathfrak{C}$ with relations

$$[A] + [C] = [B] \quad \text{for each short exact sequence} \quad 0 \to A \to B \to C \to 0.$$

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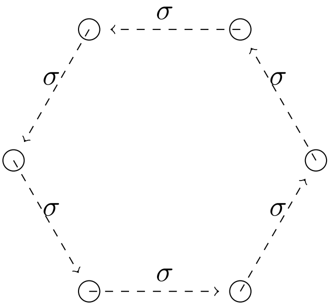
In the Grothendieck group, each module is the formal sum of its simple composition factors.

Theorem

Assume that all σ-orbits are infinite. Then in $K_0(\mathfrak{C})$ we have

$$[V_p] = [\mathrm{soc}(V_p)] + \sum_{a \in \mathfrak{m}} n_{\mathfrak{m}} L(\mathfrak{m})$$

where the occurring $L(\mathfrak{m})$ have finite support and the coefficients $n_{\mathfrak{m}} \in \mathbb{N}_0$ can be expressed combinatorially.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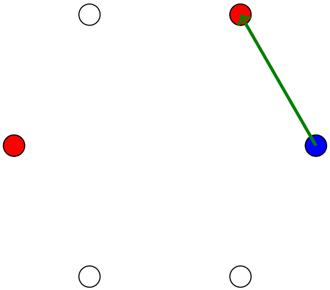

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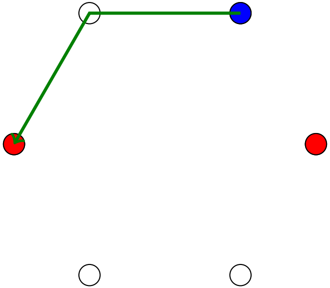

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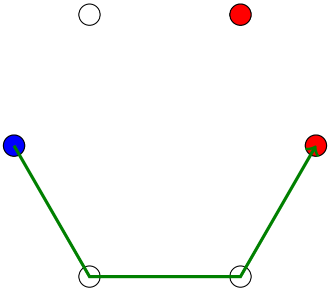

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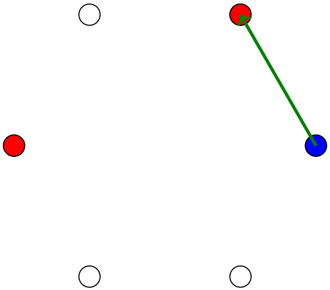

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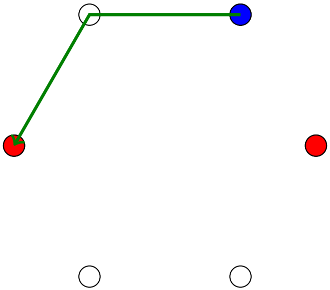

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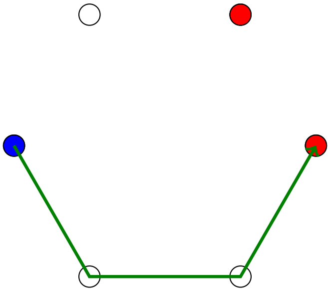

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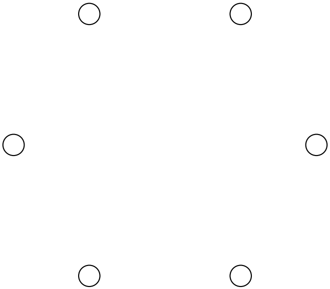

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So in this case we get an infinite 3-periodic composition series

$$\cdots \subset V_p \subset V_{p''} \subset V_{p'} \subset V_p \subset V_{p''} \subset V_{p'} \subset V_p$$

Example

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Since a has two irreducible factors there are essentially 4 modules (up to unit twists):

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So all four modules are simple unless $b \in \mathbb{Z}$, and for $b \in \mathbb{N}$ we have a composition series $\{0\} \subset V_{(h+b-2)} \subset V_{(h-b)}$ where the simple quotient is a b-dimensional simple weight module.

Thanks!