Non-associative Hilbert's basis theorems

Johan Richter

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Conventions

All rings are unital. Not neccessarily associative.

Algebraic geometry motivation

Suppose we have a set, P, of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ and we are interested in the set of common zeroes of polynomials in P.
Since $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian we can assume P is a finite set.
This is original motivation for Hilbert's basis theorem.

Theorem

If R is an associative, left (right) Noetherian ring then $R[x]$ is left (right) Noetherian.

We will describe some generalizations of this theorem.

Ore extensions

If R is an associative ring then the *Ore extension* $R[x;\sigma,\delta]$ means the associative ring generated by R and x, such that $xr=\sigma(r)x+\delta(r)$ for all $r\in R$, where σ is a endomorphism and δ satisfies two rules:

- $\delta(r+s)=\delta(r)+\delta(s)$
- $\delta(rs)=\sigma(r)\delta(s)+\delta(r)s$.

Every element of $R[x;\sigma,\delta]$ can be uniquely written as $\sum r_i x^i$ for some $r_i\in R$.

Examples

The polynomial ring $\mathbb{C}[y]$ is the Ore extension $\mathbb{C}[y; \mathrm{id}, 0]$.

The first Weyl algebra is the Ore extension $\mathbb{C}[y][x; \mathrm{id}, \frac{\partial}{\partial y}]$.

Hilbert basis theorem

The following generalization of Hilbert's basis theorem is well-known.

Theorem

If R is a an associative, left (right) Noetherian ring and σ is an automorphism of R then $R[x;\sigma,\delta]$ is left (right) Noetherian.

Non-associative Ore extensions

Let R be a non-associative ring, let σ and δ be additive maps such that $\sigma(1) = 1$ and $\delta(1) = 0$. We equip $R[X]$ with a new multiplication.

The ring structure on $R[X; \sigma, \delta]$ is defined on monomials by

$$aX^m \cdot bX^n = \sum_{i \in \mathbb{N}} a\pi_i^m(b)X^{i+n}, \tag{1}$$

for $a, b \in R$ and $m, n \in \mathbb{N}$, where π_i^m denotes the sum of all the $\binom{m}{i}$ possible compositions of i copies of σ and $m - i$ copies of δ in arbitrary order.

Non-associative Hilbert basis theorem

Proposition (Bäck and R., 2022)

Let R be a unital, non-associative ring, σ an automorphism and δ a σ-derivation on R. If R is right (left) noetherian, then so is $R[X;\sigma,\delta]$.

Quaternion example

Given a unital and associative algebra A with product $\cdot$ over a field of characteristic different from two, one may define a unital and non-associative algebra A^+ by using the *Jordan product* $\{\cdot,\cdot\}\colon A^+ \to A^+$ given by $\{a,b\} := \frac{1}{2}(a\cdot b + b\cdot a)$ for any $a,b\in A$. A^+ is then a *Jordan algebra*, i.e. a commutative algebra where any two elements a and b satisfy the *Jordan identity*, $\{\{a,b\}\{a,a\}\} = \{a,\{b,\{a,a\}\}\}$.

Example

Let σ be the automorphism on $\mathbb{H}$ defined by $\sigma(i) = -i$, $\sigma(j) = k$, and $\sigma(k) = j$. Any automorphism on $\mathbb{H}$ is also an automorphism on $\mathbb{H}^+$, and hence $\mathbb{H}^+[X;\sigma,0_{\mathbb{H}}]$ is a unital, non-associative Ore extension where e.g. $X\cdot i = -iX$, $X\cdot j = kX$, and $X\cdot k = jX$. $\mathbb{H}^+[X;\sigma,0_{\mathbb{H}}]$ is then noetherian.

Octonic example

Example

For R any non-associative ring, the *non-associative Weyl algebra over R* is the iterated, unital, non-associative Ore extension $R[Y][X; \mathrm{id}_R, \delta]$ where $\delta\colon R[Y] \to R[Y]$ is an R-linear map such that $\delta(1) = 0$. Considering $\mathbb{O}$ as a ring, the unital, non-associative Ore extension of $\mathbb{O}$ in the indeterminate Y is the unital and non-associative polynomial ring $\mathbb{O}[Y; \mathrm{id}_{\mathbb{O}}, 0_{\mathbb{O}}]$, for which we write $\mathbb{O}[Y]$. Let $\delta\colon \mathbb{O}[Y] \to \mathbb{O}[Y]$ be the $\mathbb{O}$-linear map defined on monomials by $\delta\left(aY^m\right) = maY^{m-1}$ for arbitrary $a \in \mathbb{O}$ and $m \in \mathbb{N}$, with the interpretation that $0aY^{-1}$ is 0. One readily verifies that δ is an $\mathbb{O}$-linear derivation on $\mathbb{O}[Y]$, and $\delta(1) = 0$. We thus define the *Weyl algebra over the octonions*, or the *octonionic Weyl algebra*, as $\mathbb{O}[Y][X; \mathrm{id}_{\mathbb{O}[Y]}, \delta]$ where δ is said derivation. Hence, in $\mathbb{O}[Y][X; \mathrm{id}_{\mathbb{O}[Y]}, \delta]$, $X \cdot Y = YX - 1$. The octonionic Weyl algebra is noetherian.

Hilbert' basis theorem again

Note that our assumption on σ and δ when formulating Hilbert's basis theorem mirrored the associative case, so non-associativity came only from base ring R. Can we remove that assumption?

Theorem (Bäck and R., 2024)

Let R be a unital, non-associative ring with an additive bijection σ that respects 1 and an additive map δ such that $\delta(1) = 0$. If R is right Noetherian, then so is $R[X; \sigma, \delta]$.

Proof is similar to the associative case. In associative case one can prove left case by passing to the opposite ring.

It turns out that the left version of HBT does not hold in the non-associative case.

We give a counterexample where R is a polynomial ring over a field and $R[X; \sigma, 0]$ is not left Noetherian.

Counterexample

Example

Let $R = K[Y, Z]$ where K is a field. We will define an additive bijection, σ, such that $\sigma(1) = 1$ and $\sigma(Y^i Z^j) = Y^{2i} Z^j$ if $i > 0$.

Then we set $T = R[X; \sigma, 0]$. The element Y generates an ideal I in R. We will see that the following ideal in T is not finitely generated as a left ideal:

$$J = \left\{ \sum_{i \in \mathbb{N}} r_i X^i \in T \colon r_i \in I \text{ for all } i \right\}.$$

Let us fill in some details.

Counterexample continued

Example

We first give a complete definition of the additive bijection σ. Set $U := \{1, 3, 5, \ldots\}$, and let V be the set $U \times \mathbb{N}$. Then there exist bijections $f \colon U \to \{1, 2, 3, \ldots\}$ and $g = (g_1, g_2) \colon \{2, 4, 6, \ldots\} \to V$. Define a map σ on the monomials of R as follows: $\sigma(1) = 1$, $\sigma(Y^i Z^j) = Y^{2i} Z^j$ if $i > 0$, $\sigma(Z^j) = Z^{f(j)}$ if j is odd, and $\sigma(Z^j) = Y^{g_1(j)} Z^{g_2(j)}$ if j is even. Extend σ K-linearly to all polynomials in $K[Y, Z]$. Then σ is an additive bijection that respects 1.

Counterexample continued

Example

Note that the ideal I generated by Y is mapped to the ideal generated by Y^2 by σ. Set $T := R[X;\sigma]$ and let $J = \{\sum_{i\in\mathbb{N}} r_i X^i \in T\colon r_i \in I \text{ for all } i\}$. Then J is an ideal of T. We claim that J is not finitely generated as a left ideal.

Counterexample concluded

Example

For suppose that J is generated as a left ideal by $p_1, p_2, \ldots, p_n$ for some n. Let m be the maximal degree in X of $p_1, p_2, \ldots, p_n$. Then YX^{m+1} is in the left ideal generated by these generators. So there are $s_i, t_{i,1}, t_{i,2}, \ldots \in T$ such that $YX^{m+1} = \sum_{i=1}^{n} s_i p_i + \sum_{i=1}^{n} t_{i,1}(t_{i,2} p_i) + \ldots$. There must exist terms on the right of degree at least $m+1$. Note that if a term on the right has degree $m+1$, then its coefficients belong to the ideal generated by Y^2. This would mean that the coefficient on the left of degree $m+1$ also belongs to the ideal of R generated by Y^2. This is a contradiction, so there cannot exist such a finite set of generators.

We have also proven some versions of Hilbert's basis theorem for other types of non-associative rings.

Laurent polynomial rings

A Laurent polynomial ring, $R[x^{\pm}]$, where R is an associative ring, consists of elements $\sum_{i \in \mathbb{Z}} r_i x^i$, where only finitely many r_i are non-zero, but we allow negative powers. Addition and multiplication is defined in the obvious way with x central. It is the localisation of $R[x]$ with respect to the powers of x.

Skew Laurent polynomial rings

A generalization of Laurent polynomials rings, similiar in spirit to Ore extensions, are the skew Laurent polynomial rings, $R[x^{\pm};\sigma]$. The elements are the same as in $R[x^{\pm}]$ but the multiplication rule is:

$$ax^m bx^n = a\sigma^m(b)x^{n+m}.$$

In the associative case, R is an associative ring and σ is an automorphism. We defined non-assocative Ore extensions by allowing R to be non-associative and only requiring that σ is a bijection such that $\sigma(1) = 1$.

Theorem

Let R be a unital, non-associative ring with an additive bijection σ that respects 1. If R is left (right) Noetherian, then so is $R[X^{\pm};\sigma]$.

This is a generalization of an associative result. Note no left-right assymmetry. Proof is an adaptation of a proof for group-graded rings by Bell.

We can relate the ideals of a non-associative skew Laurent polynomial ring to a subring that is a non-associative Ore extension.

Proposition

Let R be a unital, non-associative ring with an additive bijection σ that respects 1. Set $S := R[X^{\pm}; \sigma]$ and $T := R[X; \sigma]$. If I is a left ideal of S, then $I = S(I \cap T)$. If I is a right ideal of S, then $I = (I \cap T)S$.

This can be used to give an alternative proof for the right case of skew Laurent polynomial rings.

We also define non-associative skew power series rings, $R[[X;\sigma]]$, and non-associative skew Laurent series rings, $R((X;\sigma))$.

Theorem

Let R be a unital, associative ring with an additive bijection σ that respects 1. If R is right Noetherian, then so are $R[[X;\sigma]]$ and $R((X;\sigma))$.

Can one generalize the above two theorems for R non-associative?
Can one prove a left version of the above two theorems?

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