Connections in Noncommutative Riemannian Geometry

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SNAG7
Karlskrona, 2025-03-27

References

This talk is based on:

Noncommutative Riemannian geometry of Kronecker algebras
J. A., *J. Geom. Phys 199, 2024.*

On the existence of noncommutative Levi-Civita connections in derivation based calculi J. A., V. Hildebrandsson, *(In preparation.)*

Why connections?

- A connection on a differentiable manifold prescribes a way of differentiating vector fields (or, more generally, sections of a vector bundle).
- Connections play a fundamental role in differentiable geometry and can give structural information of the underlying manifold.
- General relativity is based on (pseudo-)Riemannian geometry, and geometry of space time is intimately connected to the Levi-Civita connection.
- So called gauge theories in physics use connections to describe interaction of fundamental forces.
- We would like to understand properties of a theory of connections in noncommutative geometry.
- In particular, we are interested in the existence of torsion free connections compatible with a metric, so called Levi-Civita connections.

Noncommutative differential forms

- $\mathcal{A}$ $*$-algebra over $\mathbb{C}$ (functions)
- $\mathfrak{g}$ ($*$-closed) Lie algebra of derivations on $\mathcal{A}$ (vector fields)
- $\bar{\Omega}^k_{\mathfrak{g}}$ bimodule of $Z(\mathcal{A})$-multilinear alternating maps $\omega : \mathfrak{g}^k \to \mathcal{A}$.
  (differential k-forms)

- For $\omega \in \bar{\Omega}^k_{\mathfrak{g}}$ and $\tau \in \bar{\Omega}^l_{\mathfrak{g}}$ one defines $\omega\tau \in \bar{\Omega}^{k+l}_{\mathfrak{g}}$ by antisymmetrization over the arguments.
- Exterior derivative: $d : \bar{\Omega}^k_{\mathfrak{g}} \to \bar{\Omega}^{k+1}_{\mathfrak{g}}$

$$da(\partial) = \partial a \qquad \text{for } a \in \mathcal{A} = \bar{\Omega}^0_{\mathfrak{g}}$$

$$d\omega(\partial_1, \partial_2) = \partial_1\omega(\partial_2) - \partial_2\omega(\partial_1) - \omega([\partial_1, \partial_2]) \qquad \text{for } \omega \in \bar{\Omega}^1_{\mathfrak{g}}$$

Noncommutative differential forms

- $\bar{\Omega}_{\mathfrak{g}}$ is a differential graded algebra.
- We will consider a differential subalgebra of $\bar{\Omega}_{\mathfrak{g}}$, called the restricted calculus, given by

$$\Omega_{\mathfrak{g}}^k = \{a_0 da_1 \cdots da_k : a_i \in \mathcal{A} \text{ for } i = 1, \ldots, k\}.$$

- As "metrics" on $\Omega_{\mathfrak{g}}^1$, we consider hermitian forms $h : \Omega_{\mathfrak{g}}^1 \times \Omega_{\mathfrak{g}}^1 \to \mathcal{A}$:

$$h(a\omega, \eta) = ah(\omega, \eta) \quad \text{and} \quad h(\omega, \eta)^* = h(\eta, \omega)$$

  which we often assume to be invertible in the sense that $\hat{h} : \Omega_{\mathfrak{g}}^1 \to (\Omega_{\mathfrak{g}}^1)^*$, defined by $\hat{h}(\omega)(\eta) = h(\eta, \omega)$, is a bijection.

Some notation

Definition

Let $\mathcal{A}, \mathcal{B}$ be rings and let M be a left $\mathcal{A}$-module, let N a right $\mathcal{B}$-module and let S be a $(\mathcal{A}, \mathcal{B})$-bimodule. Define

$$\mathrm{Hom}_{\mathcal{A},\mathcal{B}}(M \times N, S)$$

as the set of biadditive maps $f : M \times N \to S$ such that $f(am, nb) = af(m, n)b$ for $m \in M$, $n \in N$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Moreover, if $\mathcal{A} = \mathcal{B}$ then we write

$$\mathrm{Hom}_{\mathcal{A}}(M \times N, S) \equiv \mathrm{Hom}_{\mathcal{A},\mathcal{A}}(M \times N, S).$$

If $\mathcal{A}$ is a $*$-algebra and M is a left $\mathcal{A}$-module then $m \cdot a = a^* m$ defines a right $\mathcal{A}$-module structure on M. We denote the right $\mathcal{A}$-module obtained from M in this way by $\hat{M}$.

$\mathfrak{g}$-connections

Definition

Let $\mathcal{A}$ be a $*$-algebra and let $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ be a Lie algebra. A left $\mathfrak{g}$-connection on a left $\mathcal{A}$-module M is a map $\nabla : M \times \mathfrak{g} \to M$ such that

1. $\nabla_\partial(m + m') = \nabla_\partial m + \nabla_\partial m'$,
2. $\nabla_{\partial+\partial'} m = \nabla_\partial m + \nabla_{\partial'} m$,
3. $\nabla_\partial(am) = a(\nabla_\partial m) + (\partial a)m$,
4. $\nabla_{z\partial} m = z\nabla_\partial m$

for $m, m' \in M$, $\partial, \partial' \in \mathfrak{g}$, $a \in \mathcal{A}$, and $z \in Z(\mathcal{A})$.

Note that $\nabla \in \mathrm{Hom}_{\mathbb{C},Z(\mathcal{A})}(M \times \mathfrak{g}, M)$

Derivation based calculus

Definition

Let $\mathcal{A}$ be a $*$-algebra and let $\mathfrak{g} \subseteq \text{Der}(\mathcal{A})$ be a Lie algebra. A left $\mathcal{A}$-module M is called a left $\mathfrak{g}$-connection module if there exists a left $\mathfrak{g}$-connection $\nabla : M \times \mathfrak{g} \to M$.

The next definition introduces the noncommutative object that we think of as an analogue of a differentiable manifold.

Definition

A (left) derivation based calculus is a pair $(\mathcal{A}, \mathfrak{g})$ where $\mathcal{A}$ is a unital $*$-algebra over $\mathbb{C}$ and $\mathfrak{g} \subseteq \text{Der}(\mathcal{A})$ is a $*$-closed Lie algebra such that $\Omega^1_{\mathfrak{g}}$ is a (left) $\mathfrak{g}$-connection module.

Regular connections

Let $(\mathcal{A}, \mathfrak{g})$ be (left) derivation based calculus, and let ∇^0 be a $\mathfrak{g}$-connection on $\Omega^1_{\mathfrak{g}}$. Defining

$$(\widetilde{\nabla}_{\partial}\omega)(\partial') = (\nabla_{\partial'}\omega)(\partial) + d\omega(\partial, \partial')$$

one can check that $\widetilde{\nabla}$ satisfies the requirements of a left connection.

Moreover, it is clear that $\nabla_{\partial}\omega \in \bar{\Omega}^1_{\mathfrak{g}}$, but is not guaranteed that $\widetilde{\nabla}_{\partial}\omega \in \Omega^1_{\mathfrak{g}}$. The problem lies with

$$\eta(\partial') := (\nabla_{\partial'}\omega)(\partial)$$

considered as an element $\eta \in \bar{\Omega}^1_{\mathfrak{g}}$ for fixed $\omega \in \Omega^1_{\mathfrak{g}}$ and $\partial \in \mathfrak{g}$.

Regular connections

In light of the “problem” above, we introduce the following definition.

Definition

Let $(\mathcal{A}, \mathfrak{g})$ be a derivation based calculus and let ∇ be a connection on $\Omega^1_{\mathfrak{g}}$. The connection is called regular if the associated connection

$$(\widetilde{\nabla}_{\partial}\omega)(\partial') = (\nabla_{\partial'}\omega)(\partial) + d\omega(\partial, \partial').$$

is a (left) connection on $\Omega^1_{\mathfrak{g}}$.

We denote the set of regular connections on $\Omega^1_{\mathfrak{g}}$ by $\mathscr{C}_{\mathfrak{g}}{}^{\mathrm{Reg}}(\Omega^1_{\mathfrak{g}})$.

Torsion free connections

A connection ∇ is called *torsion free* if

$$\big(\nabla_{\partial}\omega\big)(\partial') - \big(\nabla_{\partial'}\omega\big)(\partial) = d\omega(\partial, \partial')$$

for all $\partial, \partial' \in \mathfrak{g}$ and $\omega \in \Omega^1_{\mathfrak{g}}$, which is equivalent to $\nabla = \widetilde{\nabla}$. That is, a torsion free connection is regular.

Introduce

$$\wedge, s : \mathrm{Hom}_{\mathbb{C},Z(\mathcal{A})}(\Omega^1_{\mathfrak{g}} \times \mathfrak{g}, \bar{\Omega}^1_{\mathfrak{g}}) \to \mathrm{Hom}_{\mathbb{C},Z(\mathcal{A})}(\Omega^1_{\mathfrak{g}} \times \mathfrak{g}, \bar{\Omega}^1_{\mathfrak{g}})$$

as

$$\begin{aligned}(\wedge W)(\omega, \partial_1)(\partial_2) &= W(\omega, \partial_1)(\partial_2) - W(\omega, \partial_2)(\partial_1)\\ s(W)(\omega, \partial_1)(\partial_2) &= W(\omega, \partial_1)(\partial_2) + W(\omega, \partial_2)(\partial_1)\end{aligned}$$

from which it follows that $s \circ \wedge = \wedge \circ s = 0$.

Torsion free connections

Considering $d \in \mathrm{Hom}_{\mathbb{C},Z(\mathcal{A})}(\Omega^1_{\mathfrak{g}} \times \mathfrak{g}, \bar{\Omega}^1_{\mathfrak{g}})$ via

$$d(\omega, \partial)(\partial') = d\omega(\partial, \partial'),$$

(satisfying $s \circ d = 0$ and $\wedge d = 2d$) one can write the torsion free condition as

$$d - \wedge\nabla = 0$$

now considered as an equation for maps in $\mathrm{Hom}_{\mathbb{C},Z(\mathcal{A})}(\Omega^1_{\mathfrak{g}} \times \mathfrak{g}, \bar{\Omega}^1_{\mathfrak{g}})$.

Construction of torsion free connections

Let $(\mathcal{A}, \mathfrak{g})$ be a derivation based calculus, and let ∇^0 be a *regular* $\mathfrak{g}$-connection on $\Omega^1_{\mathfrak{g}}$. We would like to construct a torsion free connection from ∇^0. To this end, define

$$\nabla = \frac{1}{2}\big(d + s(\nabla^0)\big).$$

One can easily check that ∇ satisfies the necessary conditions for a left connection and, due to the regularity of ∇^0, it is a connection on $\Omega^1_{\mathfrak{g}}$. One checks that

$$\wedge\nabla = \tfrac{1}{2} \wedge d + \tfrac{1}{2}(\wedge \circ s)(\nabla^0) = d,$$

implying that ∇ is torsion free. (In fact, all torsion free connections can be obtained in this way.)

Existence of torsion free connections

Proposition

Let $(\mathcal{A}, \mathfrak{g})$ be a derivation based calculus. There exists a torsion free connection on $\Omega^1_{\mathfrak{g}}$ if and only if ${\mathscr{C}_{\mathfrak{g}}}^{\mathrm{Reg}}(\Omega^1_{\mathfrak{g}}) \neq \emptyset$.

As we have noted, such a connection is obtained from $\nabla^0 \in {\mathscr{C}_{\mathfrak{g}}}^{\mathrm{Reg}}(\Omega^1_{\mathfrak{g}})$ as

$$\nabla = \tfrac{1}{2}\big(d + s(\nabla^0)\big).$$

Connections compatible with a hermitian form

Given a hermitian form h and $\alpha \in \mathrm{Hom}_{\mathbb{C},Z(\mathcal{A})}(\Omega^1_{\mathfrak{g}} \times \mathfrak{g}, \Omega^1_{\mathfrak{g}})$ we define $h_\alpha(\omega, \eta) \in \bar{\Omega}^1_{\mathfrak{g}}$ as

$$h_\alpha(\omega, \eta)(\partial) = h\big(\alpha(\omega, \partial), \eta\big)$$

for $\omega, \eta \in \Omega^1_{\mathfrak{g}}$ and $\partial \in \mathfrak{g}$, implying that $h_\alpha \in \mathrm{Hom}_{\mathbb{C},\mathcal{A}}(\Omega^1_{\mathfrak{g}} \times \hat{\Omega}^1_{\mathfrak{g}}, \bar{\Omega}^1_{\mathfrak{g}})$.

Moreover, introduce $* : \mathrm{Hom}_{\mathbb{C},\mathcal{A}}(\Omega^1_{\mathfrak{g}} \times \hat{\Omega}^1_{\mathfrak{g}}, \bar{\Omega}^1_{\mathfrak{g}}) \to \mathrm{Hom}_{\mathbb{C},\mathcal{A}}(\Omega^1_{\mathfrak{g}} \times \hat{\Omega}^1_{\mathfrak{g}}, \bar{\Omega}^1_{\mathfrak{g}})$

$$L^*(\omega, \eta) = L(\eta, \omega)^*$$

giving

$$h^*_\alpha(\omega, \eta)(\partial) = h\big(\omega, \alpha(\eta, \partial^*)\big)$$

Connections compatible with a hermitian form

Given a hermitian form h, a connection ∇ is said to be *compatible with h* if

$$\partial h(\omega,\eta) = h(\nabla_\partial \omega,\eta) + h(\omega,\nabla_{\partial^*}\eta)$$

for all $\partial \in \mathfrak{g}$ and $\omega,\eta \in \Omega^1_{\mathfrak{g}}$.

One can rewrite this as a relation in $\mathrm{Hom}_{\mathbb{C},\mathcal{A}}(\Omega^1_{\mathfrak{g}} \times \hat{\Omega}^1_{\mathfrak{g}}, \bar{\Omega}^1_{\mathfrak{g}})$

$$dh = h_\nabla + h^*_\nabla$$

with $dh = d \circ h$.

Connections compatible with a hermitian form

Given an arbitrary left connection ∇^0, one defines $L_{\nabla^0} : \Omega^1_{\mathfrak{g}} \times \Omega^1_{\mathfrak{g}} \to \Omega^1_{\mathfrak{g}}$ by

$$L_{\nabla^0}(\omega, \eta)(\partial) = \frac{1}{2} dh(\omega, \eta)(\partial) + \frac{1}{2}\big(h(\nabla^0_\partial \omega, \eta) - h(\omega, \nabla^0_\partial \eta)\big)$$

for $\partial \in \mathfrak{g}$, $\omega, \eta \in \Omega^1_{\mathfrak{g}}$, and sets

$$\nabla_\partial \omega = \hat{h}^{-1}\big(L_{\nabla^0}(\omega, \cdot)(\partial)^*\big).$$

One checks that ∇ is a connection compatible with h.

Connections compatible with a hermitian form

Proposition

Let $(\mathcal{A}, \mathfrak{g})$ *be a differential calculus and let* h *be an invertible hermitian form on* $\Omega^1_{\mathfrak{g}}$*. Then there exists a connection on* $\Omega^1_{\mathfrak{g}}$ *compatible with* h*.*

Summary

We introduced convenient formulations of torsion free connections:

$$\wedge\nabla - d = 0,$$

as an equation for maps in $\mathrm{Hom}_{\mathbb{C},Z(\mathcal{A})}(\Omega^1_{\mathfrak{g}} \times \mathfrak{g}, \bar{\Omega}^1_{\mathfrak{g}})$, as well as for connections compatible with hermitian forms:

$$dh = h_\nabla + h^*_\nabla$$

as maps in $\mathrm{Hom}_{\mathbb{C},\mathcal{A}}(\Omega^1_{\mathfrak{g}} \times \hat{\Omega}^1_{\mathfrak{g}}, \bar{\Omega}^1_{\mathfrak{g}})$.

Moreover, such connections are constructed by starting from an arbitrary connection ∇^0 (which needs to be regular in the torsion free case)

A natural question is if connections satisfying both conditions – so called Levi-Civita connections – exist on differential calculi. This will be the topic of Victor's talk.

Thank you!