Torsion free connections on $\mathfrak{g}$-connection modules

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Introduction

By the Gelfand-Naimark theorem, the category of (locally compact) topological spaces and the category of commutative C^*-algebras are equivalent. Similarly, we have an equivalence between the category of commutative rings and the category of affine schemes. The idea of noncommutative geometry is then to think of noncommutative algebras as corresponding to noncommutative "spaces".

In Riemannian geometry, a fundamental object of interest is the Levi-Civita connection. That is, a torsion free connection which is compatible with the Riemannian metric on the space. Every Riemannian manifold has exactly one such connection.

Background

In (commutative) geometry, a vector bundle is an assignment of a vector space to each point of a real smooth manifold M. Given a vector bundle E on M, a vector field on E is an assignment of a vector to each point in M. It can be seen as a smooth function $X : M \to E$, and the set of vector fields is denoted $\Gamma(E)$. Given two vector fields $X, Y : M \to E$ we have the $C^\infty(M)$-structure

$$(X+Y)(p) = X(p) + Y(p), \quad (fX)(p) = f(p)X(p)$$

for $f \in C^\infty(M)$, $p \in M$. Hence the space of vector fields on E is a module over the (commutative) algebra of functions. In fact, we have the following theorem by Swan (and also a similar theorem by Serre in algebraic geometry).

Theorem (R. Swan)

Let M be a compact manifold. A $C^\infty(M)$-module P is isomorphic to $\Gamma(E)$ for some vector bundle E if and only if P is a finitely generated projective module.

Background

Considering $E = TM$, then a vector field $X \in \Gamma(TM)$ can be written in local coordinates as $X = X^i \partial_i$. Then X satisfies the Leibniz rule,

$$X(fg) = f \cdot X(g) + X(f) \cdot g,$$

in other words, X is a *derivation* of the algebra $C^\infty(M)$. On a commutative space, the set of derivations defines a module. But in general, the set of derivations over a noncommutative algebra is not a module.

Dually to derivations, we have *differential forms*, which are $C^\infty(M)$ -multilinear alternating maps $\omega : \Gamma(TM) \times \cdots \times \Gamma(TM) \to C^\infty(M)$. The set Ω^k of these ω defines a module, even in the noncommutative case. Notice that, in the commutative case, $\Omega^1 = \Gamma(TM)^*$, so it is natural to consider this module in noncommutative geometry.

Background

Let $\mathcal{A}$ be a unital associative $*$-algebra over $\mathbb{C}$, and let $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ be a Lie $\mathbb{C}$-algebra closed under $*$. Let $\overline{\Omega}_{\mathfrak{g}}^{k}$ be the set of $Z(\mathcal{A})$-multilinear alternative maps

$$\omega : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k \text{ times}} \to \mathcal{A}.$$

We also let $\overline{\Omega}_{\mathfrak{g}}^{0} := \mathcal{A}$. We have a $\mathcal{A}$-bimodule structure on $\overline{\Omega}_{\mathfrak{g}}^{k}$ by

$$(a \cdot \omega \cdot b)(\partial_1, \ldots, \partial_k) := a \cdot \omega(\partial_1, \ldots, \partial_k) \cdot b, \quad a \in \mathcal{A}, \omega \in \overline{\Omega}_{\mathfrak{g}}^{k}, \partial_i \in \mathfrak{g}.$$

The *exterior derivative* $d : \overline{\Omega}_{\mathfrak{g}}^{k} \to \overline{\Omega}_{\mathfrak{g}}^{k+1}$ is given by

$$k = 0 : \quad da(\partial_1) = \partial_1 a,$$
$$k = 1 : \quad d\omega(\partial_1, \partial_2) = \partial_1 \omega(\partial_2) - \partial_2 \omega(\partial_1) - \omega([\partial_1, \partial_2])$$

for all $a \in \mathcal{A}$, $\omega \in \overline{\Omega}_{\mathfrak{g}}^{1}$, and $\partial_1, \partial_2 \in \mathfrak{g}$ (the exterior derivative can be defined for larger k, but we will only use it here for $k = 0, 1$).

$\mathfrak{g}$-connection modules

A connection on a vector bundle E is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ such that

$$\nabla_{fX} Y = f \nabla_X Y, \quad \nabla_X (fY) = f \nabla_X Y + X(f) Y.$$

for all $f \in C^\infty(M), X, Y \in \Gamma(E)$.
In the noncommutative case, we then define a connection on an A-module M (our vector bundle) as follows.

Definition ($\mathfrak{g}$-connection)

Let $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ a Lie algebra. A (left) $\mathfrak{g}$-*connection* on M is a $\mathbb{C}$-bilinear map $\nabla : \mathfrak{g} \times M \to M$ such that

$$\nabla_{z \cdot \partial} m = z \nabla_\partial m, \quad \nabla_\partial (am) = a(\nabla_\partial m) + (\partial a) m$$

for $m \in M$, $\partial \in \mathfrak{g}$, $a \in \mathcal{A}$ and $z \in Z(\mathcal{A})$.

$\mathfrak{g}$-connection modules

Definition ($\mathfrak{g}$-connection module)

Let M be a (left) $\mathcal{A}$-module and $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ a Lie algebra. We call M a $\mathfrak{g}$-*connection module* if there exists a (left) $\mathfrak{g}$-connection ∇ on M.

Denote by $\mathscr{C}_M^{\mathfrak{g}}$ the set of $\mathfrak{g}$-connections $\nabla : \mathfrak{g} \times M \to M$ on M.

Question

Which modules are $\mathfrak{g}$-connection modules?

$\mathfrak{g}$-connection modules

Cuntz and Quillen (following Connes), define a connection on a left $\mathcal{A}$-module E to be an operator $\nabla : E \to \Omega\mathcal{A} \otimes_{\mathcal{A}} E$ satisfying the Leibniz rule

$$\nabla(ae) = a \cdot (\nabla e) + (da) \cdot e, \quad e \in E, a \in \mathcal{A},$$

where $\Omega\mathcal{A}$ is the universal algebra of differential forms.

Theorem (Cuntz and Quillen, '95)

A module has a connection if and only if it is projective.

For a projective module P, a $\mathfrak{g}$-connection always exists for any $\mathfrak{g}$: Since P is a direct summand of some free module $F = \langle f^1, \ldots, f^n \rangle$, we can write $P = p(F)$, where $p : F \to F$ is a projection. On F, we define a connetion

$$\nabla_\partial(m_i f^i) := \partial(m_i) f^i.$$

Then $\tilde{\nabla} = p \circ \nabla|_{p(F)}$ is a connection on P.

$\mathfrak{g}$-connection modules

However, the converse is not always true. Consider the free noncommutative unital algebra $\mathcal{A} = \mathbb{C}\langle X_1, X_2 \rangle$ and the left ideal $I = \langle X_1 X_2 - X_2 X_1 \rangle$. Then $M = \mathcal{A}/I$ is a non-projective left $\mathcal{A}$-module. Let $\partial_{X_1}, \partial_{X_2} \in \text{Der}(\mathcal{A})$ be defined by

$$\partial_{X_1}(X_1) = \partial_{X_2}(X_2) = 1, \quad \partial_{X_1}(X_2) = \partial_{X_2}(X_1) = 0,$$

and let $\mathfrak{g}$ be the Lie algebra generated by them. Since $\partial_{X_i}(I) \subset I$, ∂_{X_i} is well-defined on M. Define $\nabla : \mathfrak{g} : M \to M$ by $\nabla_{\partial_{X_i}}[a] = [\partial_{X_i}(a)]$. Then ∇ is a connection on M. So

$$\{\text{projective modules}\} \subsetneq \{\mathfrak{g}\text{-connection modules}\}$$

Torsion free connections

Recall that $\mathscr{C}_M^{\mathfrak{g}}$ is the set of $\mathfrak{g}$-connections on M. We notice that for two connections ∇, ∇', the map $\nabla - \nabla'$ is $\mathcal{A}$-linear in M:

$$\begin{aligned}(\nabla - \nabla')(\partial, am) &= \nabla_\partial(am) - \nabla'_\partial(am) \\ &= a(\nabla_\partial m) + (\partial a)m - a(\nabla'_\partial m) - (\partial a)m \\ &= a(\nabla - \nabla')(\partial, m).\end{aligned}$$

We have the following standard proposition.

Proposition

Given a $\mathfrak{g}$-connection module M, there is a bijection between $\mathscr{C}_M^{\mathfrak{g}}$ and the set of bi-additive maps $\alpha : \mathfrak{g} \times M \to M$ such that

$$\alpha(z \cdot \partial, m) = z\alpha(\partial, m), \text{ and } \alpha(\partial, am) = a\alpha(\partial, m)$$

for $m \in M$, $\partial \in \mathfrak{g}$, $a \in \mathcal{A}$, and $z \in Z(\mathcal{A})$. We denote this set by $\mathscr{A}_M$.

Torsion free connections

Following this proposition, given $\nabla^0 \in \mathscr{C}_M^{\mathfrak{g}}$, any other connection $\nabla \in \mathscr{C}_M^{\mathfrak{g}}$ is equal to $\nabla^0 + \alpha$ for some $\alpha \in \mathscr{A}_M$.

From now onwards, consider the module $M = \overline{\Omega}_{\mathfrak{g}}^1$. For a $\mathfrak{g}$-connection ∇ on $\overline{\Omega}_{\mathfrak{g}}^1$ we define the torsion as follows.

Definition (Torsion)

The *torsion* of a left $\mathfrak{g}$-connection ∇ on $\overline{\Omega}_{\mathfrak{g}}^1$ is given by the map $T : \overline{\Omega}_{\mathfrak{g}}^1 \times \mathfrak{g} \times \mathfrak{g} \to \mathcal{A}$ defined by

$$T_\omega(\partial, \partial') = (\nabla_\partial \omega)(\partial') - (\nabla_{\partial'} \omega)(\partial) - d\omega(\partial, \partial').$$

The connection is *torsion free* if $T_\omega(\partial, \partial') = 0$ for all $\partial, \partial' \in \mathfrak{g}$ and $\omega \in \overline{\Omega}_{\mathfrak{g}}^1$.

Torsion free connections

We want to reformulate this torsion condition into something coordinate free.

Consider the wedge map $\wedge : \mathbf{Hom}_{\mathbb{C}}(\mathfrak{g} \times \overline{\Omega}^1_{\mathfrak{g}}, \overline{\Omega}^1_{\mathfrak{g}}) \to \mathbf{Hom}_{\mathbb{C}}(\overline{\Omega}^1_{\mathfrak{g}}, \overline{\Omega}^2_{\mathfrak{g}})$ where $W \in \mathbf{Hom}_{\mathbb{C}}(\mathfrak{g} \times \overline{\Omega}^1_{\mathfrak{g}}, \overline{\Omega}^1_{\mathfrak{g}})$ is sent to

$$((\wedge W)\omega)(\partial_1, \partial_2) = W(\partial_1, \omega)(\partial_2) - W(\partial_2, \omega)(\partial_1),$$

and the symmetry map $s : \mathbf{Hom}_{\mathbb{C}}(\mathfrak{g} \times \overline{\Omega}^1_{\mathfrak{g}}, \overline{\Omega}^1_{\mathfrak{g}}) \to \mathbf{Hom}_{\mathbb{C}}(\mathfrak{g} \times \overline{\Omega}^1_{\mathfrak{g}}, \overline{\Omega}^1_{\mathfrak{g}})$ where $W \in \mathbf{Hom}_{\mathbb{C}}(\mathfrak{g} \times \overline{\Omega}^1_{\mathfrak{g}}, \overline{\Omega}^1_{\mathfrak{g}})$ is sent to

$$(s(W)(\partial, \omega))(\partial') = (W(\partial, \omega))(\partial') + (W(\partial', \omega))(\partial).$$

The torsion map of ∇ can then be expressed as

$$T = \wedge\nabla - d.$$

Torsion free connections

Assuming that $\overline{\Omega}^1_{\mathfrak{g}}$ is a $\mathfrak{g}$-connection module with $\nabla^0 \in \mathscr{C}^{\mathfrak{g}}_{\overline{\Omega}^1_{\mathfrak{g}}}$, the torsion of $\nabla^\alpha := \nabla^0 + \alpha$ is

$$T = \wedge\nabla^\alpha - d = \wedge(\nabla^0 + \alpha) - d = \wedge\nabla^0 - d + \wedge\alpha,$$

and torsion freeness amounts to

$$T = 0 \iff \wedge\alpha = d - \wedge\nabla^0.$$

We notice that $d - \wedge\nabla^0 \in \mathbf{Hom}_{\mathcal{A}}(\overline{\Omega}^1_{\mathfrak{g}}, \overline{\Omega}^2_{\mathfrak{g}})$, so we want to find some $\alpha \in \mathscr{A}_{\overline{\Omega}^1_{\mathfrak{g}}}$ such that $\wedge\alpha = d - \wedge\nabla^0$.

Torsion free connections

Recall that $\mathscr{A}_{\overline{\Omega}^1_{\mathfrak{g}}}$ is the set of biadditive maps $\mathfrak{g} \times \overline{\Omega}^1_{\mathfrak{g}} \to \overline{\Omega}^1_{\mathfrak{g}}$ that is $Z(\mathcal{A})$-linear in the first variable and $\mathcal{A}$-linear in the second variable.

Proposition

The wedge map $\wedge : \mathbf{Hom}_{\mathbb{C}}(\mathfrak{g} \times \overline{\Omega}^1_{\mathfrak{g}}, \overline{\Omega}^1_{\mathfrak{g}}) \to \mathbf{Hom}_{\mathbb{C}}(\overline{\Omega}^1_{\mathfrak{g}}, \overline{\Omega}^2_{\mathfrak{g}})$ *is surjective. In particular, the restiction* $\wedge|_{\mathscr{A}_{\overline{\Omega}^1_{\mathfrak{g}}}}$ *is onto* $\mathbf{Hom}_{\mathcal{A}}(\overline{\Omega}^1_{\mathfrak{g}}, \overline{\Omega}^2_{\mathfrak{g}})$.

Proof: Let $L \in \mathbf{Hom}_{\mathbb{C}}(\overline{\Omega}^1_{\mathfrak{g}}, \overline{\Omega}^2_{\mathfrak{g}})$ and define $\alpha : \mathfrak{g} \times \overline{\Omega}^1_{\mathfrak{g}} \to \overline{\Omega}^1_{\mathfrak{g}}$ by $\alpha(\partial, \omega)(\_) := \frac{1}{2}(L\omega)(\partial, \_)$. □

Using this construction, with $L = d - \wedge\nabla^0$, we get that

$$\nabla^\alpha = \nabla^0 + \alpha = \frac{1}{2}(s(\nabla^0) + d).$$

By construction, ∇^α is a torsion free $\mathfrak{g}$-connection.

Torsion free connections

Corollary

Assume that $\overline{\Omega}^1_{\mathfrak{g}}$ is a $\mathfrak{g}$-connection module. Given a connection $\nabla^0 \in \mathscr{C}^{\mathfrak{g}}_{\overline{\Omega}^1_{\mathfrak{g}}}$, the set

$$\left\{ \frac{1}{2}\left(s(\nabla^0) + d\right) + \beta \mid \beta \in \ker \wedge \right\}$$

is the set of all torsion free connections on $\overline{\Omega}^1_{\mathfrak{g}}$.

Proof: Any $\mathfrak{g}$-connection can be written as $\frac{1}{2}(s(\nabla^0) + d) + \beta$ for some $\beta \in \mathscr{A}_{\overline{\Omega}^1_{\mathfrak{g}}}$, and it is torsion free if and only if $\wedge\beta = d - \wedge\left(\frac{1}{2}(s(\nabla^0) + d)\right)$. But we notice that

$$d - \wedge\left(\frac{1}{2}(s(\nabla^0) + d)\right) = d - \left(\frac{1}{2}(\underbrace{\wedge s}_{=0}(\nabla^0) + \underbrace{\wedge d}_{=2d})\right) = 0. \quad \square$$

Beyond?

As mentioned, Levi Civita connections are torsion free and *compatible with a metric* on the space. In noncommutative geometry we characterize this in the following way.

Definition (Hermitian form and metric compatibility)

Let M be a (left) $\mathcal{A}$-module. A (left) *hermitian form on M* is an bi-additive map $h : M \times M \to \mathcal{A}$ such that

$$h(am_1, m_2) = ah(m_1, m_2), \quad h(m_1, m_2)^* = h(m_2, m_1)$$

for all $m_1, m_2 \in M$ and $a \in \mathcal{A}$. A (left) $\mathfrak{g}$-connection ∇ on M is *compatible with a (left) hermitian form h* if

$$\partial h(m_1, m_2) = h(\nabla_\partial m_1, m_2) + h(m_1, \nabla_{\partial^*} m_2)$$

for all $m_1, m_2 \in M$ and $\partial \in \mathfrak{g}$. Also note that $\partial^*(a) = (\partial a^*)^*$ for $a \in \mathcal{A}$.

Beyond?

Question (1)

Is it possible to classify all $\mathfrak{g}$-connections compatible with a hermitian form? If so, what conditions are needed the hermitian form or on the module for there to exist a metric compatible $\mathfrak{g}$-connection?

Answer: If the hermitian form h is invertible, the set of $\mathfrak{g}$-connections compatible with h is in bijection with the set of skew-hermitian maps $\overline{\Omega}^1_{\mathfrak{g}} \times \overline{\Omega}^1_{\mathfrak{g}} \to \overline{\Omega}^1_{\mathfrak{g}}$.

Question (2)

Is it possible to construct a "noncommutative Koszul formula"; an equation which is a necessary and sufficient condition for a $\mathfrak{g}$-connection to be both torsion free and metric compatible?

Answer: Ongoing work.

Thank you!