The structure of Fell bundles

joint work with Natã Machado from UFSC

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Banach bundles

Let X be a locally compact Hausdorff space. By a *Banach bundle over X* we mean a pair $(\mathcal{B}, p)$, consisting of a Hausdorff space $\mathcal{B}$ and a continuous open surjection $p : \mathcal{B} \to X$, along with operations and norms making each *fiber* $B_x := p^{-1}(x), x \in X$, into a Banach space, and satisfying the following conditions:

1. The map $s \mapsto \|s\|$ is continuous on $\mathcal{B}$ to $\mathbb{R}$.
2. The map $(s, t) \mapsto s + t$ is continuous on $\{(s, t) \in \mathcal{B} \times \mathcal{B} : p(s) = p(t)\}$ to $\mathcal{B}$.
3. For each $\lambda \in \mathbb{C}$, the map $s \mapsto \lambda s$ is continuous on $\mathcal{B}$ to $\mathcal{B}$.
4. If $x \in X$ and $\{s_i\}$ is any net of elements in $\mathcal{B}$ such that $\|s_i\| \to 0$ and $p(s_i) \to x$, then $s_i \to 0_x$.

Banach bundles: Remarks

Condition 4. revised

Condition 4. is equivalent to the following condition: If $x \in X$, then the collection of all subsets of $\mathcal{B}$ of the form $\{s \in \mathcal{B} : p(s) \in U, \|s\| < \epsilon\}$, where U is a neighbourhood of x in X and $\epsilon > 0$ is a basis of 0_x in $\mathcal{B}$.

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Local triviality

1. Not all Banach bundles are locally trivial.

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Local triviality

1. Not all Banach bundles are locally trivial.
2. Condition 4. can be looked upon as a “fragment" of the property of local triviality.
3. It can be shown that a Banach bundle $\mathcal{B}$ whose fibers are all of the same finite dimension is necessarily locally trivial.

Fell bundles

Let (G, e) be a locally compact Hausdorff group. By a *Fell bundle over G* we mean a Banach bundle $(\mathcal{B}, p)$ over G equipped with a continuous multiplication $\cdot : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and a continuous involution $^* : \mathcal{B} \to \mathcal{B}$ satisfying the following conditions:

1. $B_g \cdot B_h \subseteq B_{gh}$ and the restriction $\cdot : B_g \times B_h \to B_{gh}$ is bilinear for all $g, h \in G$.
2. $\|st\| \leqslant \|s\|\|t\|$ for all $s, t \in \mathcal{B}$.
3. $B_g^* \subseteq B_{g^{-1}}$ and the restriction $^* : B_g \to B_{g^{-1}}$ is anti-linear for all $g \in G$.
4. $(st)^* = t^* s^*$ for all $s, t \in \mathcal{B}$.
5. $(s^*)^* = s$ for all $s \in \mathcal{B}$.
6. $\|s^* s\| = \|s\|^2$ for all $s \in \mathcal{B}$.
7. $s^* s \geqslant 0$ in B_e for all $s \in \mathcal{B}$.

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The Fell bundle is called *saturated* in case $\operatorname{span}(B_g \cdot B_h)$ is dense in B_{gh} for all $g, h \in G$.

Fell bundles: Elementary properties

Unit fiber C^*-algebra

Items 1.-6. imply that B_e is a C^*-algebra with respect to the restricted operations, the so-called *unit fiber C^*-algebra*, and item 7. refers to the standard order relation on B_e.

Fell bundles: Elementary properties

Unit fiber C*-algebra

Items 1.-6. imply that B_e is a C*-algebra with respect to the restricted operations, the so-called *unit fiber C*-algebra*, and item 7. refers to the standard order relation on B_e.

Lemma

Let (G, e) be a locally compact group and let $(\mathcal{B}, p)$ be a saturated Fell bundle over G. Then the following assertions hold:

1. B_g is a Morita equivalence B_e-bimodule for all $g \in G$.
2. $B_g \otimes_{B_e} B_h \cong B_{gh}$ as Morita equivalence B_e-bimodules for all $g, h \in G$.

Fell bundles: The Why

Why Fell bundles?

1. Fell bundles provide an important mechanism for illuminating the structure of C^*-dynamical systems and their associated crossed products.
2. Fell invented Fell bundles to understand better and extend Mackey's pioneering works in the intersection of infinite-dimensional unitary representations of groups, operator algebras, and noncommutative geometry.

Fell bundles over discrete groups

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Let θ be any real number and consider the automorphism $\alpha(f)(t) := f(t+\theta)$ of $C(\mathbb{T})$. A Fell bundles over $\mathbb{Z}$ is given by the collection $B_n := C(\mathbb{T})$, $n \in \mathbb{Z}$, along with the multiplication and involution given respectively by

$$f_m \cdot f_n := f_m \alpha^m(f_n) \qquad \text{and} \qquad f_n^* := \alpha^{-n}(\overline{f_n}) \qquad \forall m, n \in \mathbb{Z}.$$

Fell bundles over discrete groups

Example 2

Consider the $C(\mathbb{T})$-valued 2-cocycle on $\mathbb{Z}^2$ given by

$$\omega\left((m_1, n_1), (m_2, n_2)\right)(t) := \exp\left(2\pi\imath(t + n_1 m_2)\right).$$

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$$f_{(m_1,n_1)} \cdot f_{(m_2,n_2)} := f_{(m_1,n_1)} f_{(m_2,n_2)} \omega\left((m_1,n_1),(m_2,n_2)\right)$$

and

$$f^*_{(m_1,n_1)} := \omega\left((-m_1,-n_1),(m_1,n_1)\right)^* \overline{f_{(m_1,n_1)}}.$$

for all $(m_1,n_1),(m_2,n_2) \in \mathbb{Z}^2$.

From Fell bundles to C*-algebras

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- The space $\Gamma_c(G, \mathcal{B})$ of continuous, compactly supported cross sections of $(\mathcal{B}, p)$ forms a *-algebra w. r. t. the multiplication and involution given respectively by

$$(s * t)(g) := \int_G s(h)t\left(h^{-1}g\right)\, d\lambda(h) \qquad \text{and} \qquad s^*(g) := \Delta\left(g^{-1}\right) f\left(g^{-1}\right)^*$$

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- Its completion w. r. t. the L^1-norm $\|s\|_1 := \int_G \|s(g)\|\, d\lambda(g)$ yields an involutive Banach algebra, denoted $L^1(G, \mathcal{B})$.
- The enveloping C*-algebra of $L^1(G, \mathcal{B})$ is called the *cross-sectional C^*-algebra of* $(\mathcal{B}, p)$ and denoted $C^*(G, \mathcal{B})$.

Examples of cross-sectional C^*-algebras

Example 1 again

The cross-sectional C^*-algebra of the Fell bundle discussed in Example 1 is isomorphic to the quantum 2-torus $\mathbb{T}^2_\theta$.

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Fell bundle over a discrete groups vs graded algebras

A Fell bundle over a discrete group may be seen essentially as a graded algebra which has been disassembled in such a way that we are left only with the scattered resulting parts.

More examples of cross-sectional C^*-algebras

Examples

Among the examples in which the Fell bundle structure is not so obvious, lie some of the most intensely studied C^*-algebras of the past couple of decades. These include

1. group C^*-algebras,
2. quantum SU2,
3. non-commutative Heisenberg manifolds,
4. AF-algebras,
5. Cuntz-Krieger algebras,
6. and many others.

The main research objective

- Topological aspects of C^*-algebras have been extensively developed over the last decades. In contrast the study of geometric aspects of C^*-algebras is relatively new.
- One idea (of Alain Connes) is to construct the analogues of classical geometries for these noncommutative spaces by using what are called spectral triples.

The What

The overall purpose of this research project is to study the *noncommutative geometry of Fell bundles by means of spectral triples*, i. e., to construct spectral triples on the corresponding cross-sectional C^*-algebras.

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The How

The main idea is to understand better the structure of Fell bundles in terms of the underlying group and the unit standard fiber.

Intermezzo: The Picard group of a C^*-algebra

Let B be a unital C^*-algebra. The set of equivalence classes of Morita equivalence B-bimodules forms an Abelian group with respect to the internal tensor product of Hilbert B-bimodules. This group is called the *Picard group of B* and denoted $\mathrm{Pic}(B)$.

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1. For a finite-dimensional C*-algebra B, $\mathrm{Pic}(B)$ is isomorphic to the group of permutations of the spectrum of B.
2. For a compact space X, $\mathrm{Pic}\left(C(X)\right)$ is isomorphic to $\mathrm{Pic}(X) \rtimes \mathrm{Homeo}(X)$, where $\mathrm{Pic}(X)$ is the group of equivalence classes of complex line bundles over X.

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3. Let $0 < \theta < 1$ be irrational and let $\mathbb{T}^2_\theta$ be the corresponding quantum 2-torus. $\mathrm{Pic}(\mathbb{T}^2_\theta)$ is isomorphic to $\mathrm{Out}(\mathbb{T}^2_\theta)$ for quadratic θ and to $\mathrm{Out}(\mathbb{T}^2_\theta) \rtimes \mathbb{Z}$ otherwise.

An invariant for Fell bundles

Lemma

Let (G, e) be a locally compact group and let $(\mathcal{B}, p)$ be a saturated Fell bundle over G. Then the map $\varphi_{\mathcal{B}} : G \to \mathrm{Pic}(B_e)$ given by $\varphi_{\mathcal{B}}(g) := [B_g]$ is a group homomorphism.

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Proof.

By assumption, the map is well-defined and $B_g \otimes_{B_e} B_h \cong B_{gh}$ as Morita equivalence B_e-bimodules for all $g, h \in G$. It follows that

$$\varphi_{\mathcal{B}}(gh) = [B_{gh}] = [B_g \otimes_{B_e} B_h] = [B_g][B_h] = \varphi_{\mathcal{B}}(g)\varphi_{\mathcal{B}}(h) \qquad \forall g, h \in G.$$

∎

Construction and classification of Fell bundles

Let G be a locally compact group, let B be a unital C*-algebra, and let $\varphi : G \to \mathrm{Pic}(B)$ be a group homomorphism.

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A milestone

Treat the question whether there exists a saturated Fell bundle $(\mathcal{B}, p)$ over G with unit fiber C*-algebra B and $\varphi_{\mathcal{B}} = \varphi$, and, in the affirmative case, to classify all such bundles.

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Choose one representative B_g in each equivalence class $\varphi(g)$, $g \in G$, and consider the set $\mathcal{B} := \bigcup_{g \in G} B_g$ along with the natural projection map $p : \mathcal{B} \to G$.

ToDo's

1. Construct/postulate a suitable topology that turns $(\mathcal{B}, p)$ into a Banach bundle.
2. Provide suitable structure maps that turn $(\mathcal{B}, p)$ into a Fell bundle.

Thank you for your attention!