Chain conditions for graded rings

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Talk based on

- P. Lundström, *Chain conditions for rings with enough idempotents with applications to category graded rings.* Accepted for publication in Communications in Algebra.
- Available at https://arxiv.org/abs/2204.01362

Noetherian and artinian

A ring is called left/right *noetherian (artinian)* if it satisfies the *ascending (descending)* chain condition on its poset of left/right ideals.

General motivating question

- When are graded rings noetherian/artinian?

Focused motivating questions

- Noetherian/artinian rings with enough idempotents?
- Gluing of noetherian/artinian results for monoid graded rings to obtain similar results for category graded rings?
- Already done in the case of separability (SNAG 2020): J. Cala , P. Lundström and H. Pinedo, *Object-unital groupoid graded rings, crossed products and separability*, Comm. Algebra **49**:4, 1676–1696 (2021).

BACKGROUND

Noetherian results

Hilbert's basis theorem (Hilbert 1890)

If R is an associative, unital and left/right noetherian ring, then the polynomial ring $R[X]$ is left/right noetherian.

A skew Hilbert's basis theorem (Noether and Schmeidler 1920)

Suppose R is an associative and unital ring and let σ be a ring automorphism of R. If R is left/right noetherian, then the skew polynomial ring $R[X;\sigma]$ is left/right noetherian.

Noetherian results

An Ore extension Hilbert's basis theorem (Cohn 1971; Faith 1973)

Suppose R is an associative and unital ring. Let σ be a ring automorphism of R and suppose δ is a σ-derivation of R. If R is left/right noetherian, then the Ore extension $R[X;\sigma,\delta]$ is left/right noetherian.

A hom-associative Hilbert's basis theorem (Bäck and Richter 2018)

Let R be a hom-associative and unital ring with twisting map α. Let σ be a ring automorphism of R and let δ be a σ-derivation that both commute with α. Extend α homogeneously to $R[X;\sigma,\delta]$. If R is left/right noetherian, then the nonassociative Ore extension $R[X;\sigma,\delta]$ is left/right noetherian.

Noetherian results

A group G is called:

- *noetherian* if it satisfies the ascending chain condition on subgroups;
- *artinian* if it satisfies the descending chain condition on subgroups;
- *torsion-free* if the only element in G of finite order is the identity;
- *polycyclic-by-finite* if it has a finite length subnormal series

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

with each factor G_{i+1}/G_i a finite group or an infinite cyclic group.

Theorem (Folklore)

Suppose R is an associative and unital ring and let G be a group. If the group algebra $R[G]$ is left/right noetherian, then R is left/right noetherian and G is noetherian.

Theorem (Ivanov 1987)

There is a noetherian group G (or H) such that the group algebra $K[G]$ (or $K[H]$), over any associative and unital ring K, is not left (or right) noetherian.

Noetherian results

Theorem (Hall 1954)

Let R be an associative and unital ring and let G be a polycyclic-by-finite group. Then the group algebra $R[G]$ is left/right noetherian if and only if R is left/right noetherian.

Open problem (Roseblade 1973)

Is there a group G, which is not polycyclic-by-finite, and an associative and unital ring R such that the group algebra $R[G]$ is left or right noetherian?

Theorem (Strongly graded: Bell 1987. ϵ-strongly graded: Lännström 2020)

Suppose G is a polycyclic-by-finite group. Let S be an associative and unital epsilon-strongly G-graded ring with base ring R. Then S is left/right noetherian if and only if R is left/right noetherian.

Artinian results

Theorem (Connell 1963)

Suppose R is an associative and unital ring and G is a group. Then the group algebra $R[G]$ is left/right artinian if and only if R is left/right artinian and G is finite.

Theorem (Zelmanov 1977)

Suppose R is an associative and unital ring and M is a monoid. Then the monoid algebra $R[M]$ is left/right artinian if and only if R is left/right artinian and M is finite.

Theorem (Park 1979)

Suppose R is an associative and unital ring and α is a group homomorphism from a group G to $\mathrm{Aut}(R)$. Then the skew group algebra $R *_\alpha G$ is left/right artinian if and only if R is left/right artinian and G is finite.

Artinian results

Theorem (Nystedt, Öinert, Pinedo 2018)

Suppose α is a global unital action of a groupoid G on a nonassociative ring R. Then the skew groupoid algebra $R *_\alpha G$ is left/right artinian if and only if R is left/right artinian and $R_g = \{0\}$ for all but finitely many $g \in G^1$.

Theorem (Nystedt, Öinert, Pinedo 2018)

Suppose that α is a unital partial action of a groupoid G on an alternative ring R. Then the partial skew groupoid algebra $R *_\alpha G$ is left/right artinian if and only if R is left/right artinian and $R_g = \{0\}$ for all but finitely many $g \in G^1$.

Artinian group-graded rings?

- Passman has constructed an artinian twisted group ring with an infinite group.
- Thus, Connell's result does not hold for twisted group rings or, more generally, crossed products and group graded rings.
- Probably difficult in the general case!

Nevertheless

Theorem (Lännström 2020)

Suppose G is a torsion-free group. Let S be an associative and unital epsilon-strongly G-graded ring with base ring R. Then S is left/right artinian if and only if R is left/right artinian and $S_g = \{0\}$ for all but finitely many $g \in G$.

Theorem (Nastasescu and Van Oystaeyen ("Methods of graded rings"))

Suppose G is a torsion-free group. Let S be an associative and unital G-graded ring with base ring R. Then S is left/right artinian if and only if R is left/right artinian and S is finitely generated as a left/right R-module.

The rest of this presentation

- Rings with enough idempotents having a complete set of idempotents which is *strong*.
- Categories which are *hom-set strong*.
- *Hom-set-strongly* category graded rings
- Skew category algebras defined by *hom-set strong* categories.

Enough idempotents

Recall that a ring S is said to have *enough idempotents* if there exists a set $\{e_i\}_{i\in I}$ of nonzero orthogonal idempotents in S, called a *complete set of idempotents* for S, such that

$$S = \bigoplus_{i\in I} Se_i = \bigoplus_{i\in I} e_i S.$$

Given $i, j \in I$, we put

$$S_{ij} := e_i S e_j \qquad S_i := S_{ii} \qquad S_0 := \bigoplus_{i\in I} S_i$$

Strong complete set of idempotents

Suppose that S is a ring with enough idempotents.
Let $\{e_i\}_{i \in I}$ be a complete set of idempotents for S.

Proposition

The following properties are equivalent:

- $\forall (i, j, k) \in I \times I \times I$ *if two of the additive groups* S_{ij}, S_{jk} *and* S_{ik} *are nonzero, then the third one is also nonzero and* $S_{ij} S_{jk} = S_{ik}$;
- $\forall (p, q) \in I \times I$ *if one of the additive groups* S_{pq} *and* S_{qp} *is nonzero, then the other one is also nonzero and* $S_{pq} S_{qp} = S_p$;

Definition

If $\{e_i\}_{i \in I}$ satisfies any of the two equivalent properties above, then we say that $\{e_i\}_{i \in I}$ is a *strong* complete set of idempotents for S.

Theorem 1

Suppose S is a ring with enough idempotents. Let $\{e_i\}_{i \in I}$ be a complete set of idempotents for S.

- *If S is left/right artinian (noetherian), then I is finite and for every $i \in I$ the ring S_i is left/right artinian (noetherian).*
- *Suppose $\{e_i\}_{i \in I}$ is strong. Then S is left/right artinian (noetherian) if and only if I is finite and for every $i \in I$ the ring S_i is left/right artinian (noetherian).*

Example

Suppose K is a field. Let V be a nonzero K-vector space. Consider the ring

$$S = \begin{pmatrix} K & V \\ 0 & K \end{pmatrix}$$

Then S is a ring with enough idempotents where

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is a non-strong complete set of idempotents for S. The rings $S_1 \cong K$ and $S_2 \cong K$ are left/right artinian and noetherian. Let W be a K-vector subspace of V. Then

$$\begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix} \quad \text{is both a left and a right ideal of } S.$$

Therefore S is left (right) artinian/noetherian if and only if V is finite-dimensional.

Categories

- From now on, let G denote a small category (the collections of objects G_0 and morphisms G_1 are sets).
- The domain and codomain of $g \in G_1$ are denoted $d(g)$ and $c(g)$.
- We regard $G_0 \subseteq G_1$. The identity morphism $a \to a$ is denoted a.
- $G_2 := \{(g, h) \in G_1 \times G_1 \mid d(g) = c(h)\}$.
- If $(g, h) \in G_2$, then the composition of g and h is written gh.
- $G(a, b) := \{g \in G_1 \mid g : b \to a\}$ and $G(a) := G(a, a)$
- G is called a groupoid if all morphisms in G are isomorphisms.

Proposition

Suppose G is a small category. The following properties are equivalent:

- $\forall (a, b, c) \in G_0 \times G_0 \times G_0$ *if two of the sets* $G(a, b)$, $G(b, c)$ *and* $G(a, c)$ *are nonempty, then the third set is nonempty and* $G(a, b)G(b, c) = G(a, c)$*;*
- $\forall (x, y) \in G_0 \times G_0$ *if one of the sets* $G(x, y)$ *and* $G(y, x)$ *is nonempty, then the other set is nonempty and* $G(x, y)G(y, x) = G(x)$*;*

Definition

If G satisfies any of the two equivalent properties above, then we say that G is a *hom-set strong* category.

Remark

- All groupoids are hom-set strong categories.
- Not all hom-set strong categories are groupoids.

Category graded rings

- Let S be a ring which is *G-graded*. By this we mean that for every $g \in G_1$ there is an additive subgroup S_g of S such that

$$S = \bigoplus_{g \in G_1} S_g$$

and for all $g, h \in G_1$, the inclusion

$$S_g S_h \subseteq S_{gh}$$

holds, if $(g, h) \in G_2$, and $S_g S_h = \{0\}$, otherwise.
- S is called *strongly* G-graded if $S_g S_h = S_{gh}$ for all $(g, h) \in G_2$.
- If H is a subset of G, then put $S_H := \oplus_{h \in H} S_h$.

Category graded rings

- We say that the G-grading on S is *object unital* if for all $a \in G_0$ the ring S_a is unital and

$$1_{S_{c(g)}} s = s 1_{S_{d(g)}} = s$$

for all $g \in G_1$ and all $s \in S_g$.
- In that case, S is a ring with enough idempotents with

$$\{1_{S_a}\}_{a \in G_0}$$

as a complete set of idempotents.

Proposition

Suppose G is a hom-set strong category and let S be an object unital G-graded ring. Then the following properties are equivalent:

- $\forall (a, b, c) \in G_0 \times G_0 \times G_0$ *if two of the groups* $S_{G(a,b)}$, $S_{G(b,c)}$ *and* $S_{G(a,c)}$ *are nonzero, then the third is also nonzero and* $S_{G(a,b)} S_{G(b,c)} = S_{G(a,c)}$;
- $\forall (p, q) \in G_0 \times G_0$ *if one of the groups* $S_{G(p,q)}$ *and* $S_{G(q,p)}$ *is nonzero, then the other one is also nonzero and* $S_{G(p,q)} S_{G(q,p)} = S_{G(p)}$;
- *The set* $\{1_{S_a}\}_{a \in G_0}$ *is a strong complete set of idempotents for* S.

Definition

Let G be a hom-set strong category. Let S be an object unital G-graded ring. If S satisfies any of the three equivalent properties above, then we say that S is *hom-set-strongly* G-graded.

Theorem 2

Suppose S is an object unital G-graded ring.

- *If S is left/right artinian (noetherian), then G_0 is finite and, for every $a \in G_0$, the ring $S_{G(a)}$ is left/right artinian (noetherian).*
- *Suppose G is a hom-set strong category and let S be hom-set-strongly G-graded. Then S is left/right artinian (noetherian) if and only if G_0 is finite and $S_{G(a)}$ is left/right artinian (noetherian) for all $a \in G_0$.*

Proofs. This follows from Theorem 1. □

Definition

We say that a groupoid G is polycyclic-by-finite (torsion-free) if for every $a \in G_0$ the group $G(a)$ is polycyclic-by-finite (torsion-free).

Theorem 3

Suppose G is a groupoid. Let S be a ring which is G-graded and object unital.

- *Let G be polycyclic-by-finite. Then S is left/right noetherian if and only if G_0 is finite and for every $a \in G_0$ the ring S_a is left/right noetherian.*
- *Let G be torsion-free and suppose S is strongly G-graded. Then S is left/right artinian if and only if G_0 is finite and for every $a \in G_0$ the ring S_a is left/right artinian and $S_{G(a)}$ is finitely generated as a left/right S_a-module.*

Proofs. Use Theorem 2 and results for group graded rings. □

Skew category algebras

- Let G be a category and let $R = \{R_a\}_{a \in G_0}$ be a collection of unital rings.
- Let $\alpha = \{\alpha_g : R_{d(g)} \to R_{c(g)}\}_{g \in G_1}$ be a collection of ring isomorphisms.
- We say that α is a *skew category system* if α is a functor $G \to \text{Ring}$.
- We say that the associated *skew category algebra* $R *_\alpha G$ is the set of formal finite sums of elements of the form rg for $r \in R_{c(g)}$ and $g \in G_1$.
- Addition in $R *_\alpha G$ is defined by the relations

$$rg + r'g = (r + r')g \quad \text{for } r, r' \in R_{c(g)} \text{ and } g \in G_1.$$

- Multiplication in $R *_\alpha G$ is defined by the additive extension of

$$rg \cdot r'h = r\alpha_g(r')gh \quad \text{for } r \in R_{c(g)}, r' \in R_{c(h)}, \text{ when } (g, h) \in G_2$$

and $rg \cdot r'h = 0$, when $(g, h) \notin G_2$.

Skew category algebras

- $R *_\alpha G$ is G-graded if we put $(R *_\alpha G)_g = R_{c(g)}g$ for $g \in G_1$.
- With this grading $R *_\alpha G$ is strongly G-graded.
- The set $\{1_{R_a} a\}_{a \in G_0}$ is a complete set of idempotents for $R *_\alpha G$.
- If G is a groupoid (group, monoid), then $R *_\alpha G$ is called a skew groupoid (group, monoid) algebra.
- If all the rings in R coincide with a ring T and the ring isomorphisms in α are identity maps, then $R *_\alpha G$ is called a *category algebra* and is denoted by $T[G]$.
- In that case, if G is groupoid (group, monoid), then $T[G]$ is called a groupoid (group, monoid) algebra.

Proposition

The set $\{1_{R_a} a\}_{a \in G_0}$ is a strong complete set of idempotents for $R *_\alpha G$ if and only if the category G is hom-set strong. In that case, $R *_\alpha G$ is hom-set-strongly G-graded.

Theorem 4

Suppose G is hom-set strong. Then the skew category algebra $R *_\alpha G$ is left/right artinian (noetherian) if and only if G_0 is finite and for every $a \in G_0$ the skew monoid algebra $R_a *_{\alpha|_{G(a)}} G(a)$ is left/right artinian (noetherian).

Proofs. This follows from Theorem 2. □

Corollary

- *Suppose G is a groupoid. Then the skew groupoid algebra $R *_\alpha G$ is left/right artinian if and only if G is finite and every R_a, for $a \in G_0$, is left/right artinian.*
- *Suppose G is hom-set strong and let T be a unital ring. Then the category algebra $T[G]$ is left/right artinian if and only if G is finite and T is left/right artinian.*

Proofs. This follows from Theorem 4 and results by Park and Zelmanov. □

Thank you for your attention!