The Ext-algebra of Standard Modules of Twisted Doubles

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Conventions

Let $\mathbb{k}$ be a fixed algebraically closed field.

Let Λ be a finite-dimensional $\mathbb{k}$-algebra.

Throughout, we will mainly work with finite-dimensional left modules.

Let $S(1), \ldots, S(n)$ be a complete list of non-isomorphic simple Λ-modules.

For each $S(i)$, let $P(i)$ be the projective cover of S_i.

Standard Modules

Let $\leq$ be a partial ordering on the set $1, \ldots, n$.

Definition

The i'th standard module of Λ with respect to $\leq$ is given by the quotient:

$$\Delta(i) := P(i) \Big/ \sum_{\substack{f : P(j) \to P(i) \\ i \not\leq j}} \mathrm{im}(f)$$

We write $\mathcal{F}(\Delta)$ for the full subcategory of Λ-mod consisting of those modules M that admit filtrations:

$$0 = M_0 \subset M_1 \subset \ldots \subset M_m = M$$

with $M_{i+1}/M_i \cong \Delta(a_i)$ for some $1 \leq a_i \leq n$.

Quasi-Hereditary Algebras

Definition
A pair $(\Lambda, \leq)$ is said to be **quasi-hereditary** if the following conditions are met:

(i) ${}_{\Lambda}\Lambda \in \mathcal{F}(\Delta)$

(ii) $\mathrm{End}_{\Lambda}(\Delta(i)) \cong \Bbbk$ for all i.

Examples include:

- Algebras associated to blocks of category $\mathcal{O}$ of a semi-simple complex Lie algebra.
- All finite-dimensional algebras Λ with $\mathrm{gl.dim}(\Lambda) \leq 2$.
- Blocks of Schur algebras.

An algebra is said to be **directed** if it quasi-hereditary and all standard modules are simple.

Exact Borel Subalgebras

Assume that $(\Lambda, \leq)$ is quasi-hereditary.

Definition

A subalgebra $B \subset \Lambda$ is called an **exact Borel subalgebra** provided that the following conditions are met:

(i) B has the same number of isoclasses of simple modules as Λ, and we fix an indexing $S_B(1), \ldots, S_B(n)$ of the simple B-modules.

(ii) $(B, \leq)$ is directed.

(iii) The functor $\Lambda \otimes_B - : B\text{-mod} \to \Lambda\text{-mod}$ is exact.

(iv) $\Lambda \otimes_B S_B(i) \cong \Delta(i)$ for all $1 \leq i \leq n$.

Observe that the image of $\Lambda \otimes_B -$ is contained in $\mathcal{F}(\Delta)$.

Δ-Subalgebras

There is also a kind of dual notion to Borel subalgebras:

Definition

A subalgebra $A \subset \Lambda$ is called a **Δ-subalgebra** provided that the following conditions are met:

(i) A has the same number of isoclasses of simple modules as Λ, and we fix an indexing $S_A(1), \ldots, S_A(n)$ of the simple A-modules.

(ii) $(A, \geq)$ is directed.

(iii) $\mathrm{Res}_A^\Lambda(\Delta(i)) \cong P_A(i)$ for all $1 \leq i \leq n$.

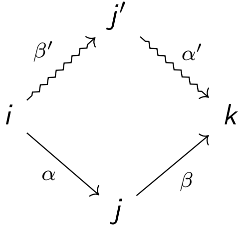
They are dual in the sense that Λ admits an exact Borel subalgebra if and only if $\Lambda^{\mathbf{op}}$ admits a Δ-subalgebra.

Twisted Doubles I: Twisting Pairs

Let Q and Q' be two quivers with equal vertex sets.

Definition

Let $\beta : j \to k$ be an arrow in Q and let $\alpha : i \to j$ be an arrow in Q'. A **twisting pair** of (β, α) is a diagram:



where α' is a path in Q' and β' is a path in Q. We write $\mathrm{Tw}(\beta, \alpha)$ for the set of all twisting pairs of β and α.

Twisted Doubles II: Twisting Relations

A **labeling** on (Q, Q') is the choice of a function

$$M_{\beta\alpha} : \mathrm{Tw}(\beta, \alpha) \to \Bbbk$$

for each pair of arrows $\beta : j \to k$ in Q and $\alpha : i \to j$ in Q'.

The values $M_{\beta\alpha}(\alpha', \beta')$ are called **twisting constants.**

We can form a new quiver $Q \sqcup Q'$ by:

- The vertices are the vertices of Q ($=$ vertices of Q').
- The set of arrows is the disjoint union of the arrow sets of Q and Q'.

Twisted Doubles III: The Construction

Let $B = \Bbbk Q/I$ and $A = \Bbbk Q'/I'$ where Q and Q' have equal vertex sets. Let M be a labeling on (Q, Q').

Definition (König and Xi, 1998)

The **twisted double** of B and A with respect to M is given by the path algebra of the quiver $Q \sqcup Q'$ modulo the ideal generated by:

- All the relations in the ideals I and I'.
- For each $\beta : j \to k$ in Q and $\alpha : i \to j$ in Q', take the **twisting relation**:

$$\beta\alpha = \sum_{(\alpha',\beta') \in \mathrm{Tw}(\beta,\alpha)} M_{\beta\alpha}(\alpha', \beta') \cdot \alpha'\beta'$$

We write $\mathcal{A}(B, A, M)$ for this algebra.

Twisted Doubles IV: An Example

Let $B = \Bbbk\mathbb{A}_4$ and let $A = B^{\mathbf{op}}$.

Let $M = \mathbf{1}$ be the labeling on $(\mathbb{A}_4, \mathbb{A}_4^{\mathbf{op}})$ in which all twisting constants are equal to one.

Then $\mathcal{A}(\Bbbk\mathbb{A}_4, \Bbbk\mathbb{A}_4^{\mathbf{op}}, \mathbf{1})$ is given by the quiver:

$$1 \underset{\alpha_1}{\overset{\beta_1}{\rightleftarrows}} 2 \underset{\alpha_2}{\overset{\beta_2}{\rightleftarrows}} 3 \underset{\alpha_3}{\overset{\beta_3}{\rightleftarrows}} 4$$

with relations:

- $\beta_1\alpha_1 = \alpha_2\beta_2 + \alpha_2\alpha_3\beta_3\beta_2$
- $\beta_2\alpha_2 = \alpha_3\beta_3$
- $\beta_3\alpha_3 = 0$

Twisted Doubles V: Quasi-Heredity

Let $\leq$ be a partial order on the vertices so that $(B, \leq)$ and $(A, \geq)$ are directed. Let M be a labeling on (Q, Q').

Question: Is $(\mathcal{A}(B, A, M), \leq)$ quasi-hereditary with exact Borel subalgebra B and Δ-subalgebra A?

By some general results due to [König, 1995], the answer to our question is yes if and only if the multiplication map:

$$\mu : A \otimes_S B \to \mathcal{A}(B, A, M)$$

is an isomorphism in S-mod-S, where S denotes the semi-simple subalgebra generated by the vertex idempotents.

Examples

Whether or not μ is an isomorphism has been resolved in the following situations:

- If $B = \Bbbk Q$ and $A = \Bbbk Q'$ and M is any labeling, then μ is an isomorphism. Proven in part by Deng and Xi in 1995.
- If M is the labeling in which all twisting constants are zero then μ is an isomorphism. Proven by Deng and Xi in 1993.

Theorem (Norlén Jäderberg)

If B and A are monomial algebras, then there is a complete description of when μ is an isomorphism.

Remark. Not much is known about more general situations.

The Ext-algebra of Standard Modules

For a quasi-hereditary algebra, an important problem is understanding the category $\mathcal{F}(\Delta)$.

If Λ possesses an exact Borel subalgebra, then the functor $\Lambda \otimes_B - : B\text{-mod} \to \mathcal{F}(\Delta)$ enables the study of $\mathcal{F}(\Delta)$ in terms of B-modules.

Problem: Not every quasi-hereditary algebra admits an exact Borel subalgebra.

The Ext-algebra of Standard Modules

Theorem (König, Külshammer, Osvienko, 2014)

Every quasi-hereditary algebra is Morita equivalent to a quasi-hereditary algebra with an exact Borel subalgebra.

The algebra $\mathrm{Ext}^*(\Delta, \Delta)$ plays a crucial role in the construction of the new algebra.

In addition, the exact Borel subalgebra obtained in this way satisfies some additional regularity conditions that makes it extremely well-behaved.

This motivates the study of $\mathrm{Ext}^*(\Delta, \Delta)$ even for quasi-hereditary algebras where an exact Borel subalgebra is already present.

Main Theorem

Theorem (Norlén Jäderberg)

- *Let* $B = \mathbb{k}Q/I$ *and* $A = \mathbb{k}Q'/I'$ *where* Q *and* Q' *have equal vertex sets.*
- *Let* M *be a labeling on* (Q, Q')*.*
- *Let* $\leq$ *be a partial order so that* $(\mathcal{A}(B, A, M), \leq)$ *is quasi-hereditary,* B *is an exact Borel subalgebra and* A *is a* Δ*-subalgebra.*

Then there is an isomorphism of graded algebras:

$$\mathrm{Ext}^*_{\mathcal{A}(B,A,M)}(\Delta, \Delta) \cong \mathcal{A}(\mathrm{Ext}^*_B(\mathbb{S}, \mathbb{S}), A^{\mathbf{op}}, \widehat{M})$$

for some labeling $\widehat{M}$*. Here* $\mathbb{S} := \bigoplus_{i=1}^n S_B(i)$*.*

Some Remarks

Remark 1. Thuresson proved the formula in the special case where M is the zero labeling in 2022.

Remark 2. The induced labeling $\widehat{M}$ need not be unique as it depends heavily on the choice of bases in $A^{\mathbf{op}}$ and $\mathrm{Ext}_B^*(\mathbb{S}, \mathbb{S})$. Very little is known about when two different labelings give isomorphic twisted doubles.

Remark 3. The hardest step in applying our formula is finding a labeling $\widehat{M}$. However, in the case where B and A are monomial algebras, one can use the combinatorics of Anick chains to arrive at a nice formula for $\widehat{M}$.

Example

Recall the twisted double $\mathcal{A}(\Bbbk\mathbb{A}_4, \Bbbk\mathbb{A}_4^{\mathsf{op}}, \mathbf{1})$ from before.

A standard computation gives

$$\mathrm{Ext}^*_{\Bbbk\mathbb{A}_4}(\mathbb{S}, \mathbb{S}) \cong \Bbbk\mathbb{A}_4/\mathrm{rad}(\Bbbk\mathbb{A}_4)^2$$

and using the formula from the previous theorem, it can be shown that:

$$\mathrm{Ext}^*_{\mathcal{A}(\Bbbk\mathbb{A}_4, \Bbbk\mathbb{A}_4^{\mathsf{op}}, \mathbf{1})}(\Delta, \Delta) \cong \mathcal{A}(\Bbbk\mathbb{A}_4/\mathrm{rad}(\Bbbk\mathbb{A}_4)^2, \Bbbk\mathbb{A}_4, \mathbf{1})$$

Example

In other words, $\mathrm{Ext}^*_{\mathcal{A}(\Bbbk\mathbb{A}_4, \Bbbk\mathbb{A}_4^{\mathsf{op}}, \mathbf{1})}(\Delta, \Delta)$ is given by the quiver:

$$1 \underset{\alpha_1}{\overset{\beta_1}{\rightrightarrows}} 2 \underset{\alpha_2}{\overset{\beta_2}{\rightrightarrows}} 3 \underset{\alpha_3}{\overset{\beta_3}{\rightrightarrows}} 4$$

with relations

- $\beta_2\beta_1 = 0$
- $\beta_3\beta_2 = 0$
- $\beta_2\alpha_1 = \alpha_2\beta_1$
- $\beta_3\alpha_2 = \alpha_3\beta_2$

Future Research: Twisted Tensor Products

Recall that the multiplication map

$$\mu : A \otimes_S B \to \mathcal{A}(B, A, M)$$

is an isomorphism in S-mod-S. In the proof of our formula for $\mathrm{Ext}^*(\Delta, \Delta)$, it is also shown that:

$$\mathrm{Ext}^*_{\mathcal{A}(B,A,M)}(\Delta, \Delta) \cong A^{\mathbf{op}} \otimes_S \mathrm{Ext}^*_B(\mathbb{S}, \mathbb{S})$$

in S-mod-S.

Moreover, the proof never really made use of the directedness of B and A, as well as many of the properties of quasi-hereditary algebras.

Twisted Tensor Products: An idea

This suggests that our formula is really just a special case of a much more general phenomenon.

Idea

Let S be an algebra and let A and B be two algebra objects in S-mod-S. Given a bimodule morphism:

$$\tau : B \otimes_S A \to A \otimes_S B$$

subject to some associativity and unitality axioms, we can define a multiplication on $A \otimes_S B$ by:

$$A \otimes_S B \otimes_S A \otimes_S B \xrightarrow{1_A \otimes \tau \otimes 1_B} A \otimes_S A \otimes_S B \otimes_S B \xrightarrow{m_A \otimes m_B} A \otimes_S B$$

This defines the structure of a unital algebra on $A \otimes_S B$.

Example

As an example, for $\mathcal{A}(B, A, M)$, the morphism τ corresponds to the composite:

$$B \otimes_S A \xrightarrow{\mu'} \mathcal{A}(B, A, M) \xrightarrow{\mu^{-1}} A \otimes_S B$$

where μ' is another multiplication map.

On arrows, this takes the form:

$$\tau(\beta \otimes \alpha) := \sum_{(\alpha', \beta') \in \mathrm{Tw}(\beta, \alpha)} M_{\beta\alpha}(\alpha', \beta') \cdot (\alpha' \otimes \beta')$$

Conjecture

Question/Conjecture: Under which hypotheses on A, B, S, and τ, is there a morphism:

$$\widehat{\tau} : \mathrm{Ext}^*_B(\mathbb{S}, \mathbb{S}) \otimes_S A^{\mathbf{op}} \to A^{\mathbf{op}} \otimes_S \mathrm{Ext}^*_B(\mathbb{S}, \mathbb{S})$$

so that there is an isomorphism of graded algebras:

$$\mathrm{Ext}^*_{A \otimes_S B}(A \otimes_S \mathbb{S}, A \otimes_S \mathbb{S}) \cong A^{\mathbf{op}} \otimes_S \mathrm{Ext}^*_B(\mathbb{S}, \mathbb{S})$$

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