# The noncommutative geometry of frame bundles

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## Conventions (I)

- The letters $P$ and $X$ stand for locally compact spaces.
- The letter $G$ stands for a compact group.
- All representations of $G$ are assumed to be finite-dimensional and unitary.
- We denote a representation $\sigma : G \to \mathrm{U}(V_\sigma)$ by the pair $(\sigma, V_\sigma)$ or simply by $\sigma$ when no ambiguity is possible.
- We write $\mathrm{Rep}(G)$ for the representation category of $G$.

A continuous action $r : P \times G \to P$ is called a *(topological) principal bundle* if $r$ is free, i. e., all stabilizer groups are trivial.

**The Hopf fibration**

The canonical right action of $\mathbb{S}^1$ on $\mathrm{SU}(2)$ is a principal bundle. Moreover, $\mathrm{SU}(2)/\mathbb{S}^1 \cong \mathbb{S}^2$. This yields the famous Hopf fibration

$$\mathbb{S}^1 \hookrightarrow \mathrm{SU}(2) \cong \mathbb{S}^3 \twoheadrightarrow \mathbb{S}^2.$$

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Let $r : P \times G \to P$ be a principal bundle.

- For an object $\sigma$ in $\mathrm{Rep}(G)$ we obtain a hermitian vector bundle over $P/G$ by putting $\Gamma_P(\sigma) := (P \times V_\sigma)/G$.
- For a morphism $T$ in $\mathrm{Rep}(G)$ we obtain a bundle map of $\Gamma_P(\sigma)$ by putting $\Gamma(T)([(p, v]) := ([(p, T(v)])$.
- We thus get a functor $\Gamma_P : \mathrm{Rep}(G) \to \mathrm{Vec}_\mathrm{h}(P/G)$, $\mathrm{Vec}_\mathrm{h}(P/G)$ being the category of hermitian vector bundles over $P/G$.
- It is possible to reconstruct $P$ from $F_P$ up to isomorphism (Tannaka-Krein duality theory).

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Let $q : E \to X$ be a hermitian vector bundle with typical fibre $V$. The *frame bundle*

$$\mathrm{Fr}(E) := \bigsqcup_{x \in X} \mathrm{Iso}\,(V, E_x)\,, \qquad E_x := q^{-1}(\{x\}),$$

carries the structure of a principal $\mathrm{U}(V)$-bundle w. r. t. the canonical right action of $\mathrm{U}(V)$ on $\mathrm{Fr}(E)$.

**Remark**

- Let $(\pi, V)$ be the standard representation $(\pi, V)$ of $\mathrm{U}(V)$. Then $\Gamma_{\mathrm{Fr}(E)}(\pi) = (\mathrm{Fr}(E) \times V)/\mathrm{U}(V) \cong E$.
- If $E$ is real and orientable, then we may consider the reduction of $\mathrm{Fr}(E)$ to $\mathrm{SO}(n)$ given by orientation-preserving isometries.

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# Why frame bundles?

- Frame bundles constitute a key tool for studying vector bundles.
- The frame bundle can be utilized in order to attach several new vector bundles in a functorial manner.
- A connection on the frame bundle induces covariant derivatives on all associated bundles in a coherent way, leading to a various important geometric constructions, i. e.:
  - In Riemannian geometry, the Levi-Civita connection on $\mathrm{Fr}(TX)$, $X$ being a manifold, induces a covariant derivative on the tensor fields, leading, for instance, to the Riemannian curvature of $X$.
  - In Riemannian spin geometry, a "spin connection" on $\mathrm{Fr}(TX)$, $X$ being a manifold, induces a covariant derivative on the spinor bundle, leading to the Dirac operator on the the spinor bundle.

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## The What

Find an algebraic analogue of the geometric inducing procedure for frame bundles in the setting of $C^*$-algebraic principal bundles.

## The Why

This is part of a larger program with the purpose to give a novel bundle-theoretic perspective on NC Riemannian spin geometry:

- Lift Dirac operators to noncommutative principal bundles.
- Propose a notion of noncommutative frame bundles.
- Extend noncommutative principal bundles by central extensions.
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## Conventions (II)

- The letters $\mathcal{A}$ and $\mathcal{B}$ stand for unital C$^*$-algebras.
- We use $\mathrm{Irr}(G)$ to denote the set of equivalence classes of irreducible representations.
- We write $\mathrm{Corr}(\mathcal{B})$ for the category of correspondences over $\mathcal{B}$.

## $C^*$-dynamical systems

- By a *$C^*$-dynamical system* we mean a triple $(\mathcal{A}, G, \alpha)$ with a strongly continuous group homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{A})$.
- We typically write $\mathcal{B}$ for the corresponding fixed point algebra, i. e., $\mathcal{B} := \mathcal{A}^G := \{x \in \mathcal{A} : (\forall g \in G)\, \alpha_g(x) = x\}$.
- Like every representation of $G$, $\mathcal{A}$ can be decomposed into its isotypic components $A(\sigma) := P_\sigma(\mathcal{A})$, $\sigma \in \mathrm{Irr}(G)$, where

$$P_\sigma(x) := \dim(\sigma) \cdot \int_G \mathrm{Tr}(\sigma_g^*)\, \alpha_g(x)\, dg, \qquad x \in \mathcal{A}.$$

That is, $\bigoplus^{\mathrm{alg}}_{\sigma \in \mathrm{Irr}(G)} A(\sigma)$ is a dense $^*$-subalgebra of $\mathcal{A}$.

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- Note that $\Gamma_{\mathcal{A}}(\mathbb{1}) = \mathcal{B}$, $\mathbb{1}$ being the trivial representation of $G$.
- Each $\Gamma_{\mathcal{A}}(\sigma)$ is naturally a correspondence over $\mathcal{B}$ w. r. t.
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  - the restriction of $\langle a \otimes v, b \otimes w \rangle_{\mathcal{A}} := \langle v, w \rangle a^* b$ to $\mathcal{B}$.
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- We thus get a linear functor $\Gamma_{\mathcal{A}} : \mathrm{Rep}(G) \to \mathrm{Corr}(\mathcal{B})$ such that:
  (i) $\Gamma_{\mathcal{A}}(\mathbb{1}) = \mathcal{B}$.
  (ii) $\Gamma_{\mathcal{A}}(T)^* = \Gamma_{\mathcal{A}}(T^*)$ for all morphisms $T$ in $\mathrm{Rep}(G)$.
- We have natural $\mathcal{B}$-bilinear isometries

$$m_{\mathcal{A}}(\sigma,\tau) : \Gamma_{\mathcal{A}}(\sigma) \otimes_{\mathcal{B}} \Gamma_{\mathcal{A}}(\tau) \to \Gamma_{\mathcal{A}}(\sigma \otimes \tau), \quad x \otimes y \mapsto x_{12} y_{13}$$

  for all objects $\sigma, \tau$ in $\mathrm{Rep}(G)$ such that:
  (iii) $m_{\mathcal{A}}(\sigma, \tau \otimes \rho)\,(\mathrm{id} \otimes_{\mathcal{B}} m_{\mathcal{A}}(\tau,\rho)) = m_{\mathcal{A}}(\sigma \otimes \tau, \rho)\,(m_{\mathcal{A}}(\sigma,\tau) \otimes_{\mathcal{B}} \mathrm{id})$ for all objects $\sigma, \tau, \rho$ in $\mathrm{Rep}(G)$.

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  (i) $\Gamma_{\mathcal{A}}(\mathbb{1}) = \mathcal{B}$.
  (ii) $\Gamma_{\mathcal{A}}(T)^* = \Gamma_{\mathcal{A}}(T^*)$ for all morphisms $T$ in $\mathrm{Rep}(G)$.
- We have natural $\mathcal{B}$-bilinear isometries

$$m_{\mathcal{A}}(\sigma,\tau) : \Gamma_{\mathcal{A}}(\sigma) \otimes_{\mathcal{B}} \Gamma_{\mathcal{A}}(\tau) \to \Gamma_{\mathcal{A}}(\sigma \otimes \tau), \quad x \otimes y \mapsto x_{12} y_{13}$$

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# Associated vector bundles (II)

- We thus get a linear functor $\Gamma_{\mathcal{A}} : \mathrm{Rep}(G) \to \mathrm{Corr}(\mathcal{B})$ such that:
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## Weak unitary tensor functors (Neshveyev, [1, Def. 2.1])

By a *weak unitary tensor functor* $\mathrm{Rep}(G) \to \mathrm{Corr}(\mathcal{B})$ we mean

- a linear functor $\Gamma : \mathrm{Rep}(G) \to \mathrm{Corr}(\mathcal{B})$ together with
- natural $\mathcal{B}$-bilinear isometries

$$m(\sigma, \tau) : \Gamma(\sigma) \otimes_{\mathcal{B}} \Gamma(\tau) \to \Gamma(\sigma \otimes \tau)$$

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**Theorem (Neshveyev, [1, Thm. 2.3])**

There is a 1:1 correspondence between C*-dynamical systems $(\mathcal{A}, G, \alpha)$ and weak unitary tensor functors $\mathrm{Rep}(G) \to \mathrm{Corr}(\mathcal{B})$.

A C*-dynamical system $(\mathcal{A}, G, \alpha)$ is called *free* if the *Ellwood map*

$$\Phi : \mathcal{A} \otimes_{\text{alg}} \mathcal{A} \to C(G, \mathcal{A}), \quad \Phi(x \otimes y)(g) := x\alpha_g(y)$$

has dense range (w. r. t. the canonical C*-norm on $C(G, \mathcal{A})$).

**Remark (A characterization of freeness)**

A C*-dynamical system $(\mathcal{A}, G, \alpha)$ is free if and only if the induced maps $m_{\mathcal{A}}(\sigma, \tau)$ for all objects $\sigma, \tau$ in $\operatorname{Rep}(G)$ are unitary. That is, there is a 1:1 correspondence between free C*-dynamical systems $(\mathcal{A}, G, \alpha)$ and unitary tensor functors $\operatorname{Rep}(G) \to \operatorname{Corr}(\mathcal{B})$.

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A C$^*$-dynamical system $(\mathcal{A}, G, \alpha)$ is called *free* if the *Ellwood map*

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A C$^*$-dynamical system $(\mathcal{A}, G, \alpha)$ is free if and only if the induced maps $m_{\mathcal{A}}(\sigma, \tau)$ for all objects $\sigma, \tau$ in $\mathrm{Rep}(G)$ are unitary. That is, there is a 1:1 correspondence between free C$^*$-dynamical systems $(\mathcal{A}, G, \alpha)$ and unitary tensor functors $\mathrm{Rep}(G) \to \mathrm{Corr}(\mathcal{B})$.

Let $r : P \times G \to P$ be a continuous action on a compact space $P$. The map $\alpha : G \to \mathrm{Aut}(C(P))$ given by

$$\alpha_g(f)(p) := f(r(p, g))$$

yields a C*-dynamical system $(C(P), G, \alpha)$. Moreover, TFAE:

- $r$ is free in the classical sense, i. e., a principal bundle.
- $P \times G \to P \times P$, $(p, g) \mapsto (p, r(p, g))$ is injective.
- $(C(P), G, \alpha)$ is free in the sense of Ellwood.

Note: We have $C(P)^G \cong C(P/G)$.

## Classical actions

Let $r : P \times G \to P$ be a continuous action on a compact space $P$. The map $\alpha : G \to \mathrm{Aut}(C(P))$ given by

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Note: We have $C(P)^G \cong C(P/G)$.

- Each *quantum n-torus* $\mathbb{T}^n_\theta$ together with its natural $\mathbb{T}^n$-action. We have ${\mathbb{T}^n_\theta}^{\mathbb{T}^n} = \mathbb{C}$.
- Each crossed product $\mathcal{A} \rtimes_\alpha D$ with discrete Abelian $D$ together with its natural dual action by $\hat{D}$. We have $(\mathcal{A} \rtimes_\alpha D)^{\hat{D}} = \mathcal{A}$.
- $C^*(H_3)$ of the discrete, 3-dim. *Heisenberg group* $H_3$ together with its (fibrewise) $\mathbb{T}^2$-action. We have $C^*(H_3)^{\mathbb{T}^2} \cong C(\mathbb{T})$.
- *Woronowicz's quantum* SU(2) together with its natural gauge action. We have $\mathrm{SU}_q(2)^{\mathbb{T}} = S^2_q$. (*Quantum Hopf fibration*).
- For a graph $\Gamma$ with finitely many vertices the gauge action on $C^*(\Gamma)$ is free if and only if $\Gamma$ is row-finite and has no sinks.
- The *Connes-Landi sphere* $\mathbb{S}^7_\theta$ admits an action by SU(2) which gives a free C*-dynamical system with fixed point algebra $\mathbb{S}^4_{\theta'}$.

# Why studying free actions?

- The study and classification of actions of (quantum-) groups on C*-algebras is intrinsically interesting.
- Free actions are closely related to saturated Fell bundles, Hopf-Galois extensions, and strongly graded rings.
- NCPB's are becoming increasingly prevalent in applications to analysis, geometry, and mathematical physics, e. g.,:
  - Unbounded $KK$-theory
  - Noncommutative Riemannian geometry
  - Noncommutative gauge theory
  - TQTF's and T-duality

# Table of Contents

# The problem (again)

**Problem (Noncommutative frame bundles**

Find an algebraic analogue of the geometric inducing procedure for frame bundles within the framework of free C$^*$-dynamical systems.

**How?**

Given a certain correspondence over $\mathcal{B}$ which plays the role of the vector bundle associated with an ordinary frame bundle w. r. t. the standard representation $\pi$ of $SO(n)$, $n \geq 3$, let's say $M$, we provide a construction procedure, by means of unitary tensor functors, for a free C$^*$-dynamical system $(\mathcal{A}_M, SO(n), \alpha_M)$ with $\Gamma_{\mathcal{A}_M}(\pi) \cong M$.

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## The central notion (W' 23)

Let $(\pi, V)$ be the standard representation of $\mathrm{SO}(n)$, $n \geq 3$. We say that *a correspondence $M$ over $\mathcal{B}$ is tensorial of type $\pi$* if there exist injective linear maps

$$\varphi_{k,l} : C_{k,l} := \mathrm{Hom}_{\mathrm{SO}(n)}\left(V^{\otimes k}, V^{\otimes l}\right) \to \mathcal{L}\left(M^{\otimes k}, M^{\otimes l}\right)$$

for all $k, l \geq 0$ such that:

(C) $\varphi_{l,m}(T')\varphi_{k,l}(T) = \varphi_{k,m}(T'T)$,

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## The main idea of the construction (I)

Let $M$ be a correspondence over $\mathcal{B}$ that is tensorial of type $\pi$.

- The main idea is to put together a unitary tensor functor $\mathrm{Rep}(\mathrm{SO}(n)) \to \mathrm{Corr}(\mathcal{B})$ by means of $M$.
- In particular, we need "building blocks", i. e., a correspondence over $\mathcal{B}$ for each $\sigma \in \mathrm{Irr}(\mathrm{SO}(n))$.
- Crucially, each irreducible representation of $\mathrm{SO}(n)$ occurs as a subrepresentation of some $(\pi^{\otimes k}, V^{\otimes k})$, $k \geq 0$.
- For each $\sigma \in \mathrm{Irr}(\mathrm{SO}(n))$ we choose a representative $(\sigma, V_\sigma)$ that is a subrepresentation of some $(\pi^{\otimes k}, V^{\otimes k})$, $k \geq 0$.
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- Clearly, $P_\sigma \in C_{k,k}$, and hence $\varphi_{k,k}(P_\sigma)$ acts as an adjointable operator on $M^{\otimes k}$.
- (C) and (A) combined imply that $\varphi_{k,k}(P_\sigma)$ is a projection, and from this it may be concluded that

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**Theorem (W' 23)**

Each correspondence $M$ over $\mathcal{B}$ that is tensorial of type $\pi$ gives rise to a unitary tensor functor $\Gamma_M : \mathrm{Rep}(\mathrm{SO}(n)) \to \mathrm{Corr}(\mathcal{B})$ such that $\Gamma_M(\pi) = M$, and, in consequence, to a free C$^*$-dynamical system $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ such that $\Gamma_{\mathcal{A}_M}(\pi) \cong \Gamma_M(\pi) = M$.

**Theorem (W' 23)**

The construction recovers the classical setting of frame bundles.

**Remark/Definition**

The latter result justifies to call the free C$^*$-dynamical system $(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)$ from the former result the *noncommutative frame bundle associated with $M$*.

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# The main results (II)

**Corollary (W' 23)**

The map $[(\mathcal{A}, \mathrm{SO}(n), \alpha)] \mapsto [\Gamma_{\mathcal{A}}(\pi)]$ yields a 1:1 correspondence between equivalence classes of free C$^*$-dynamical systems with structure group $\mathrm{SO}(n)$ and fixed point algebra $\mathcal{B}$ and equivalence classes of correspondences over $\mathcal{B}$ that are tensorial of type $\pi$. Its inverse is given by $[M] \mapsto [(\mathcal{A}_M, \mathrm{SO}(n), \alpha_M)]$.

Thank you for your attention!

[1] S. Neshveyev. “Duality theory for nonergodic actions”. In: *Münster J. Math.* 7.2 (2013), pp. 414–437. ISSN: 1867-5778; 1867-5786/e.