Hom-algebra structures and twisted derivations

AMS MSC 2020: **17B61 Hom-Lie and related algebras,**

17D30 (non-Lie) Hom algebras and topics

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Outline

Motivation

- discretizations (quasi-deformations) of vector fields, (quantum) q-deformations of finite-dimensional Lie algebras and infinite-dimensional Lie algebras as Witt and Virasoro algebras; q-deformed vertex operator models of CFT; quantum field theory; quantization;
- q-deformed Heisenberg (Weyl) algebras, quantum oscillator algebras, quantum algebras, braided Lie algebras;
- q-analysis, q-special functions;
- q-deformations of differential and homological algebra, non-commutative twisted differential calculi and geometry;
- color Lie algebras and superalgebras (Γ-graded ϵ-Lie);
- quantum physics and quantum field theory;
- non-associative algebras;
- non-commutative geometry and non-commutative analysis;
- non-commutative probability and stochastic processes

σ-derivations

(twisted, deformed, discretized derivations)

$\mathcal{A}$ – (commutative) associative $\mathbb{K}$-algebra with unity
$\sigma : \mathcal{A} \to \mathcal{A}$ algebra endomorphism
σ-**derivations**

- $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ linear map
- twisted (deformed) Leibniz rule

$$\partial_\sigma(a \cdot b) = \partial_\sigma(a) \cdot b + \sigma(a) \cdot \partial_\sigma(b)$$

Examples of σ-derivations

- q-difference operator
  $(\partial\, a)(x) = a(qx) - a(x)$
  $(\partial\, ab)(x) = (\partial\, a)(x)b(x) + a(qx)(\partial\, b)(x)$
  $\sigma(a)(x) = a(qx)$
- Jackson q-derivative
  $(\partial\, a)(x) = (D_q a)(x) = \frac{a(qx)-a(x)}{qx-x}$
  $(\partial\, ab)(x) = (\partial a)(x)b(x) + a(qx)(\partial b)(x)$
  $\sigma(a)(x) = a(qx), \quad \lim_{q\to 1} D_q(a)(x) = a'(x)$
- **General twisted difference operators)**
  $\Omega \subset \mathbb{K}$ any subset of a field, $\quad T : \Omega \to \Omega$
  Any transformation without fixed points in Ω
  A any algebra of functions a on Ω such that
  $\sigma(a)(x) = a(T(x)) \in A$
  $\partial_\sigma : a(x) \mapsto \frac{a(T(x))-a(x)}{T(x)-x} = \left(\frac{(\sigma-id)}{(T-id)}a\right)(x)$
  $\partial_\sigma(a\cdot b) = \partial_\sigma(a)\cdot b + \sigma(a)\cdot\partial_\sigma(b)$

σ-derivations on UFD

Hartwig, Larsson, Silvestrov, Deformations of Lie algebras using σ-derivations,

J. of Algebra, 295 (2006), 314-361, Preprint Institute Mittag-Lefler, **2003**, Preprint Lund University **2003**

Theorem 1 $\mathcal{A}$ is UFD (unique factorization domain)

$$\Downarrow$$

Space of all σ-derivations $\mathfrak{D}_\sigma(\mathcal{A})$ is a free rank one $\mathcal{A}$-module with generator

$$\Delta = \frac{(\mathrm{id} - \sigma)}{g} : \quad a \longmapsto \frac{(\mathrm{id} - \sigma)(a)}{g}$$

where $g = \gcd\big((\mathrm{id} - \sigma)(\mathcal{A})\big)$

Quasi-Hom-Lie algebras of twisted (deformed) vector fields

Hartwig, Larsson, Silvestrov, Deformations of Lie algebras using σ-derivations,

J. of Algebra, 295 (2006), 314-361, Preprint Institute Mittag-Lefler, **2003**, Preprint Lund University **2003**

$\mathcal{A}$ commutative algebra, $\sigma : \mathcal{A} \to \mathcal{A}$ algebra endomorphism, $\Delta : \mathcal{A} \to \mathcal{A}$ σ-derivation,

$Ann(\Delta) = \{a \in A \mid a\Delta = 0\}$, σ-twisted vector fields $\mathcal{A} \cdot \Delta$

Theorem 2 Bracket on $\mathcal{A} \cdot \Delta$ (well-defined if $\sigma(Ann(\Delta)) \subseteq Ann(\Delta)$)

$$\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = (\sigma(a) \cdot \Delta)(b \cdot \Delta) - (\sigma(b) \cdot \Delta)(a \cdot \Delta)$$

Closure $\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = (\sigma(a)\Delta(b) - \sigma(b)\Delta(a)) \cdot \Delta$

Skew-symmetry $\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = -\langle b \cdot \Delta, a \cdot \Delta \rangle_\sigma$

Twisted 6 term Jacobi Identity

If $\Delta \circ \sigma(a) = \delta \cdot \sigma \circ \Delta(a)$, for some $\delta \in A$, then $\forall a, b, c \in \mathcal{A}$:

$$\sum_{\circlearrowleft_{a,b,c}} \Big(\big\langle \sigma(a)\cdot\Delta, \langle b\cdot\Delta, c\cdot\Delta\rangle_\sigma \big\rangle_\sigma + \delta\cdot\big\langle a\cdot\Delta, \langle b\cdot\Delta, c\cdot\Delta\rangle_\sigma \big\rangle_\sigma \Big) = 0$$

$\mathcal{A}$ is UFD $\Rightarrow$ $\delta = \frac{\sigma(g)}{g}$, $g = GCD(id - \sigma)(\mathcal{A})$

Quasi-Lie algebra

Quasi-Lie algebras were first introduced in

D. Larsson, S. Silvestrov, Quasi-Lie algebras, In: Jurgen Fuchs, et al. (eds), "Noncommutative Geometry and Representation Theory in Mathematical Physics", American Mathematical Society, Contemporary Mathematics, Vol. 391, 2005.

$(L, \langle \cdot, \cdot \rangle_L, \alpha, \beta, \omega, \theta)$

1. L is a linear space over $\mathbb{F}$,
2. $\langle \cdot, \cdot \rangle_L : L \times L \to L$ is a bilinear product or bracket in L;
3. $\alpha, \beta : L \to L$, are linear maps,
4. $\omega : D_\omega \to \mathcal{L}_{\mathbb{F}}(L)$ and $\theta : D_\theta \to \mathcal{L}_{\mathbb{F}}(L)$ are maps with domains of definition $D_\omega, D_\theta \subseteq L \times L$,

ω-**Symmetry** $\forall (x, y) \in D_\omega$

$$\langle x, y \rangle_L = \omega(x, y) \langle y, x \rangle_L,$$

Quasi-Jacobi identity $\forall (z, x), (x, y), (y, z) \in D_\theta$

$$\circlearrowleft_{x,y,z} \left\{ \theta(z, x) \big(\langle \alpha(x), \langle y, z \rangle_L \rangle_L + \beta \langle x, \langle y, z \rangle_L \rangle_L \big) \right\} = 0$$

Hom-Lie algebra

Hom-Lie algebras is special subclass of Quasi-Lie algebras

$$\beta = 0, \quad \omega = -\operatorname{id}_L, \quad \theta = \operatorname{id}_L$$

1. linear map $\alpha : L \to L$
2. bilinear multiplication (bracket) $\langle \cdot, \cdot \rangle_\alpha$ such that

- **skew-symmetry** $\langle x, y \rangle_\alpha = -\langle y, x \rangle_\alpha$
- **Hom-Lie Jacobi identity** $\forall x, y, z \in L$

$$\sum_{\circlearrowleft_{x,y,z}} \big\langle \alpha(x), \langle y, z \rangle_\alpha \big\rangle_\alpha = 0$$

Quasi-Hom-Lie algebras

$(L, \langle\cdot,\cdot\rangle_L, \alpha, \beta, \omega)$

1. L is a linear space over field $\mathbb{K}$
2. $\langle\cdot,\cdot\rangle_L : L \times L \to L$ is a bilinear map
3. $\alpha, \beta : L \to L$ are linear maps
4. $\omega : D_\omega \to End(L)$ is a map with domain of definition $D_\omega \subseteq L \times L$ taking values in linear operators on L

- (β-**twisting**) α is a β-twisted algebra endomorphism:

$$\langle \alpha(x), \alpha(y)\rangle_L = \beta \circ \alpha \langle x, y\rangle_L \quad \forall x, y \in L$$

- (ω-**symmetry**) $\langle x, y\rangle_L = \omega(x, y)\langle y, x\rangle_L \quad \forall (x, y) \in D_\omega$
- **Quasi-Hom-Lie Jacobi identity** $\forall (z, x), (x, y), (y, z) \in D_\omega$

$$\circlearrowleft_{x,y,z} \left\{ \omega(z, x)\Big(\langle \alpha(x), \langle y, z\rangle_L\rangle_L + \beta\langle x, \langle y, z\rangle_L\rangle_L\Big)\right\} = 0$$

q-Deformed Witt algebra Witt_q

$$\mathfrak{D}_\sigma(\mathcal{A}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot d_n$$

$\Delta = tD_q = \frac{\sigma - \mathrm{id}}{q-1} : f(t) \mapsto \frac{f(qt)-f(t)}{q-1}$

$\sigma(t) = qt, \quad \sigma(f)(t) = f(qt), \quad \{n\}_q = \frac{q^n-1}{q-1}$

Skew-symmetric product. $d_n = -t^n\Delta$

$$\langle d_n, d_m\rangle = q^n d_n d_m - q^m d_m d_n = \left(\{n\}_q - \{m\}_q\right) d_{n+m}$$

Graded Hom-Lie algebra $\langle L_n, L_m\rangle \subseteq L_{n+m}$

Hom-Lie algebra Jacobi-identity

$$\sum_{\circlearrowleft_{n,m,l}} \left\langle (q^n+1)d_n, \langle d_m, d_l\rangle\right\rangle = 0$$

$$\sum_{\circlearrowleft_{n,m,l}} \left\langle \alpha(d_n), \langle d_m, d_l\rangle\right\rangle = 0$$

$\alpha(d_n) = (q^n+1)d_n$

q-Deformed Virasoro algebra. Hom-Lie central extension

$$(\mathrm{Vir}_q, \hat{\sigma}) = (\mathrm{Witt}_q \oplus \mathbb{C}\cdot \mathbf{c}, \hat{\sigma}) \quad \{d_n : n \in \mathbb{Z}\} \cup \{\mathbf{c}\}$$

$$\hat{\sigma} : \mathrm{Vir}_q \to \mathrm{Vir}_q, \quad \hat{\sigma}(d_n) = q^n d_n, \quad \hat{\sigma}(\mathbf{c}) = \mathbf{c}$$

$$\langle d_n, d_m \rangle = \left(\{n\}_q - \{m\}_q\right) d_{n+m} + \\ + \delta_{n+m,0} \frac{q^{-n}}{6(1+q^n)} \{n-1\}_q \{n\}_q \{n+1\}_q \mathbf{c}$$

$$\langle \mathbf{c}, \mathrm{Vir}_q \rangle = 0$$

Loop quasi Hom-Lie algebras

Quasi-Hom-Lie algebra $\mathfrak{g}$

$$\Downarrow$$

Linear space

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{K}[t, t^{-1}]$$

The algebra of Laurent polynomials with coefficients in the qhl-algebra $\mathfrak{g}$.

$\alpha_{\hat{\mathfrak{g}}} := \alpha_{\mathfrak{g}} \otimes \mathrm{id}$

$\beta_{\hat{\mathfrak{g}}} := \beta_{\mathfrak{g}} \otimes \mathrm{id}$

$\omega_{\hat{\mathfrak{g}}} := \omega_{\mathfrak{g}} \otimes \mathrm{id}$

$\langle x \otimes t^n, y \otimes t^m \rangle_{\hat{\mathfrak{g}}} = \langle x, y \rangle_{\mathfrak{g}} \otimes t^{n+m}$

$\hat{\mathfrak{g}}$ is a quasi Hom-Lie algebra.

Non-linear Quasi-Lie deformations of Witt algebra

$$\mathfrak{D}_\sigma(\mathcal{A}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot d_n,$$

$$D = \alpha t^{-k+1}\frac{\mathrm{id}-\sigma}{t-qt^s}, \quad \sigma(t) = qt^s$$

Skew-symmetric product $\qquad d_n = -t^n D$

$$\langle d_n, d_m\rangle_\sigma = q^n d_{ns} d_m - q^m d_{ms} d_n$$

$$\langle d_n, d_m\rangle_\sigma = \text{linear combinations of generators}$$

For $n, m \geq 0$:

$$\langle d_n, d_m\rangle_\sigma = \alpha \mathrm{sign}(n-m) \sum_{l=\min(n,m)}^{\max(n,m)-1} q^{n+m-1-l} d_{s(n+m-1)-(k-1)-l(s-1)}$$

Non-linear Quasi-Lie deformations of Witt algebra

σ-deformed Jacobi-identity

$$\sum_{\circlearrowleft_{n,m,l}} \left(\left\langle q^n d_{ns}, \langle d_m, d_l \rangle_\sigma \right\rangle_\sigma + \underbrace{q^k t^{k(s-1)} \sum_{r=0}^{s-1} (qt^{s-1})^r}_{=\delta} \left\langle d_n \langle d_m, d_l \rangle_\sigma \right\rangle_\sigma \right) = 0.$$

Quasi-Hom-Lie algebra, not Hom-Lie algebra for $s \neq 1$

Other non-linear Quasi-Lie deformations of Witt algebra

$\mathcal{A} = \mathbb{C}[t, t^{-t}]$, $\sigma(t) = qt^s$, $s \in \mathbb{Z}$:
$D = \frac{\mathrm{id} - \sigma}{\eta^{-1} \cdot t^k}$ generates cyclic $\mathcal{A}$-submodule $\mathfrak{M}$ of $\mathfrak{D}_\sigma(\mathcal{A})$, proper for $s \neq 1$ (for $s = 1$: $\sigma(t) = \beta t$ for some $\beta \in \mathbb{K}$)

$$D \circ \sigma = q^k t^{(s-1)k} \, \sigma \circ D = \delta \, \sigma \circ D$$

Theorem The linear space $\mathfrak{M} = \bigoplus_{i \in \mathbb{Z}} \mathbb{K} \cdot d_i$, $d_i = -t^i D$ is a quasi-Lie algebra ($\eta \in \mathbb{C}$)

$$\langle d_n, d_m \rangle_\sigma = q^n d_{ns} d_m - q^m d_{ms} d_n = \eta q^m d_{ms+n-k} - \eta q^n d_{ns+m-k}$$

σ-deformed Jacobi identity

$$\sum_{\circlearrowleft_{n,m,l}} \Big(\big\langle q^n d_{ns}, \langle d_m, d_l \rangle_\sigma \big\rangle_\sigma + \underbrace{q^k t^{(s-1)k}}_{=\delta} \big\langle d_n, \langle d_m, d_l \rangle_\sigma \big\rangle_\sigma \Big) = 0$$

$q = 1$, $k = 0$ **and** $s = 1$
yields a commutative algebra with countable number of generators instead of the Witt algebra.

Non-linear Quasi-Lie deformations of Witt algebra are almost graded

Almost graded algebras $s = 1$:
$\langle L_n, L_m \rangle_\sigma \subseteq L_{n+m-k}$
(quasi-Lie deformations of) Krichever-Novikov type algebras,
Graded $k = 0, s = 1$: $\langle L_n, L_m \rangle_\sigma \subseteq L_{n+m}$
Hyper almost Graded algebras:

$$\langle L_n, L_m \rangle_\sigma \subseteq \bigoplus_{j \in \mathbb{Z} \cap [ms+n-k, ns+m-k]} L_j$$

$ms + n - k = m + n + m(s-1) - k$

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$

- D. Larsson, S. Silvestrov, Quasi-deformations of $sl_2(\mathbb{F})$ using twisted derivations, Communications in Algebra, **35**, 4303-4318, (2007). arXiv:math/0506172 [math.RA] (2005).
- D. Larsson, S. Silvestrov, The Lie algebra $sl_2(F)$ and quasi-deformations, Czechoslovak Journal of Physics, **55**, 11 (2005), 1467-1472.
- D. Larsson, G. Sigurdsson, S. Silvestrov, On some almost quadratic algebras coming from twisted derivations, J. Nonlin. Math. Phys. Vol. 13, (2006). (Preprints in mathematical sciences (2006:9), LUTFMA-5073-2006, Centre for Mathematical Sciences, Lund University. 11 pp.)
- D. Larsson, S. D. Silvestrov, Quasi-deformations of $sl_2(F)$ with base $\mathbb{R}[t, t^{-1}]$. Czechoslovak J. Phys. 56 (2006), no. 10-11, 1227–1230

$$\mathfrak{sl}_2(\mathbb{K}) : [\mathbf{h}, \mathbf{e}] = 2\mathbf{e},\ [\mathbf{h}, \mathbf{f}] = -2\mathbf{f},\ [\mathbf{e}, \mathbf{f}] = \mathbf{h}$$

Representation

$\mathbf{e} \mapsto \partial$, $\mathbf{h} \mapsto -2t\partial$, $\mathbf{f} \mapsto -t^2\partial$

Lie algebra product $[a, b] = ab - ba$

σ-Twisted vector fields

$\mathbf{e} \mapsto \partial_\sigma$, $\mathbf{h} \mapsto -2t\partial_\sigma$, $\mathbf{f} \mapsto -t^2\partial_\sigma$

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$

$$\begin{aligned}\langle a\cdot\Delta, b\cdot\Delta\rangle_\sigma &= (\sigma(a)\cdot\Delta)(b\cdot\Delta)-(\sigma(b)\cdot\Delta)(a\cdot\Delta)\\ &= (\sigma(a)\Delta(b)-\sigma(b)\Delta(a))\cdot\Delta\end{aligned}$$

Assumption $\sigma(1)=1$ and $\partial_\sigma(1)=0$

$$\Downarrow$$

$$\begin{aligned}\langle \mathbf{h},\mathbf{f}\rangle &= 2\sigma(t)\partial_\sigma(t)t\partial_\sigma\\ \langle \mathbf{h},\mathbf{e}\rangle &= 2\partial_\sigma(t)\partial_\sigma\\ \langle \mathbf{e},\mathbf{f}\rangle &= -(\sigma(t)+t)\partial_\sigma(t)\partial_\sigma\end{aligned}$$

Closure of the bracket on $L=\mathbb{K}\mathbf{e}\bigoplus\mathbb{K}\mathbf{f}\bigoplus\mathbb{K}\mathbf{h}$

$$\Updownarrow$$

$$\deg\sigma(t)\partial_\sigma(t)t\leq 2$$

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$. Affine $\sigma(t)$ and ∂_σ is $c\frac{d}{dx}$-like on t^k

Case 1: $\mathcal{A} = \mathbb{K}[t]$, $\sigma(t) = q_0 + q_1 t$, $\partial_\sigma(t) = p_0$

$$\langle \mathbf{h}, \mathbf{f} \rangle : \ -2q_0\mathbf{ef} + q_1\mathbf{hf} + q_0^2\mathbf{eh} - q_0q_1\mathbf{h^2} - q_1^2\mathbf{fh} = -q_0p_0\mathbf{h} - 2q_1p_0\mathbf{f}$$

$$\langle \mathbf{h}, \mathbf{e} \rangle : \ -2q_0\mathbf{e^2} + q_1\mathbf{he} - \mathbf{eh} = 2p_0\mathbf{e}$$

$$\langle \mathbf{e}, \mathbf{f} \rangle : \ \mathbf{ef} + q_0^2\mathbf{e^2} - q_0q_1\mathbf{he} - q_1^2\mathbf{fe} = -q_0p_0\mathbf{e} + \frac{q_1+1}{2}p_0\mathbf{h}.$$

$q_1 = 1$, $q_0 = p_0 = 0$ gives $\mathfrak{sl}_2(\mathbb{K})$

${}_q\mathfrak{sl}_2(\mathbb{K})$ Jackson $\mathfrak{sl}_2(\mathbb{K})$ (quasi-Lie algebra). Linear $\sigma(t)$ and ∂_σ is $c\frac{d}{dx}$-like on t^k

$q_0 = 0,\ q = q_1 \neq 0\ (\frac{d}{dt} \mapsto D_q)$

$$\mathbf{hf} - q\mathbf{fh} = -2p_0\mathbf{f}$$
$$\mathbf{he} - q^{-1}\mathbf{eh} = 2q^{-1}p_0\mathbf{e}$$
$$\mathbf{ef} - q^2\mathbf{fe} = \frac{q+1}{2}p_0\mathbf{h}$$

Iterated Ore extension of $\mathbb{K}[z]$, Auslander-regular, global dimension at most three, has PBW-basis, noetherian domain of GK-dimension three, Koszul as an almost quadratic algebra

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$

$p_0 = 0 \to$ "abelianized" version. But $\partial_\sigma = 0$
$p_0 = 1$:
$\langle \mathbf{h}, \mathbf{f} \rangle = -2q\mathbf{f}$, $\langle \mathbf{h}, \mathbf{e} \rangle = 2\mathbf{e}$, $\langle \mathbf{e}, \mathbf{f} \rangle = \frac{q+1}{2}\mathbf{h}$

Twisted (Hom-Lie) Jacobi identity

$\alpha(\mathbf{e}) = \frac{q^{-1}+1}{2}\,\mathbf{e}, \quad \alpha(\mathbf{h}) = \mathbf{h}, \quad \alpha(\mathbf{f}) = \frac{q+1}{2}\,\mathbf{f}$

$$\langle \alpha(\mathbf{e}), \langle \mathbf{f}, \mathbf{h} \rangle \rangle + \langle \alpha(\mathbf{f}), \langle \mathbf{h}, \mathbf{e} \rangle \rangle + \langle \alpha(\mathbf{h}), \langle \mathbf{e}, \mathbf{f} \rangle \rangle = 0$$

Quasi-Lie deformations of $\mathfrak{sl}_2(\mathbb{K})$ on $\mathbb{K}[t]/(t^3)$

$\mathcal{A} = \mathbb{K}[t]/(t^3)$, $\sigma(t) = q_1 t + q_2 t^2$, $\partial_\sigma(t) = p_1 t$.

$$\begin{aligned}
\langle h, f\rangle : &\quad q_1 hf + 2q_2 f^2 - q_1^2 fh = 0 \\
\langle h, e\rangle : &\quad q_1 he + 2q_2 fe - eh = -p_1 h - 2p_2 f \\
\langle e, f\rangle : &\quad ef - q_1^2 fe = p_1(q_1 + 1)f.
\end{aligned}$$

$\mathfrak{sl}_2(\mathbb{K})$ cannot be recovered in any "limit"

$p_1 = 0$ representation collapse $\partial_\sigma(t) = 0$

Quasi-Lie quasi-deformations of $\mathfrak{sl}_2(\mathbb{K})$ on $\mathbb{K}[t]/(t^3)$. Special limits

"non-commutative deformation of $\mathbb{K}[x,y]$ in f-direction"

$q_1 = 1, q_2 = -\frac{1}{2}, p_1 = p_2 = 0$

$\mathbf{hf} - \mathbf{fh} = \mathbf{f}^2, \quad \mathbf{he} - \mathbf{eh} = \mathbf{fe}, \quad \mathbf{ef} - \mathbf{fe} = 0$

$\mathbf{f} = 0 \to \mathbb{K}[h, e]$

solvable 3-dimensional Lie algebra

$q_1 = 1,\ q_2 = 0,\ p_1 = 1,\ p_2 = a/2$

$\mathbf{hf} - \mathbf{fh} = 0, \mathbf{he} - \mathbf{eh} = -\mathbf{h} - a\mathbf{f}, \mathbf{ef} - \mathbf{fe} = 2\mathbf{f}$

Heisenberg Lie algebra $p_1 = 0, p_2 = -1/2$

$\mathbf{hf} - \mathbf{fh} = 0, \mathbf{he} - \mathbf{eh} = \mathbf{f}, \mathbf{ef} - \mathbf{fe} = 0$

Polynomials in 3 commuting variables

$q_1 = 1,\ q_2 = 0,\ p_1 = p_2 = 0 \to \mathbb{K}[x, y, z]$

Quasi-Hom-Lie algebra Jacobi identity. Case $q_1 p_1 \neq 0$

$$\sum_{\circlearrowleft_{x,y,z}} (\langle \sigma(x), \langle y, z\rangle\rangle + \underbrace{(1 - \frac{q_1 p_2 - p_2 - p_1 q_2}{p_1} t + \xi_2 t^2)}_{=\delta} \langle x, \langle y, z\rangle\rangle) = 0$$

Hom-associative algebras $\mapsto$ Hom-Lie algebras

Hom-associative algebras were first introduced in

Makhlouf A., Silvestrov S.D., Hom-algebra structures, J. Gen. Lie Theory, Appl. 2 (2), 51–64 (2008).

(Preprints in Mathematical Sciences 2006:10, LUTFMA-5074-2006, Centre for Mathematical Sciences, Lund University, **2006**)

Hom-associative algebra (V, μ, α)

V linear space,

$\mu : V \times V \to V$ bilinear map,

$\alpha : V \to V$ linear map (linear space homomorphism)

Hom-associativity (two notations for multiplication)

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))$$

$$\boxed{\alpha(x)(yz) = (xy)\alpha(z)}$$

$\alpha = Id_V \Leftrightarrow$ associative algebra

Hom-associative algebras $\mapsto$ Hom-Lie algebras

Theorem

Hom-associative algebras are **Hom-Lie admissible**.

(V, μ, α) is Hom-associative algebra $\alpha(x)(yz) = (xy)\alpha(z)$

$$\Downarrow \qquad [\cdot,\cdot]_\alpha = xy - yx$$

$$(V, [\cdot,\cdot]_\alpha, \alpha) \text{ is Hom-Lie algebra}$$

- **skew-symmetry** $\boxed{[x,y]_\alpha = -[y,x]_\alpha}$
- **Hom-Lie Jacobi identity** $\forall x, y, z \in L$

$$\sum_{\circlearrowleft (x,y,z)} \left[\alpha(x), [y,z]_\alpha\right]_\alpha =$$

$$\boxed{\left[\alpha(x), [y,z]_\alpha\right]_\alpha + \left[\alpha(z), [x,y]_\alpha\right]_\alpha + \left[\alpha(y), [z,x]_\alpha\right]_\alpha = 0}$$

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

$G \subseteq \mathcal{S}_3$ subgroup of the permutations group
$(-1)^{\varepsilon(s)}$ is the signature of the permutation
$s(x_1, x_2, x_3) = (x_{s(1)}, x_{s(2)}, x_{s(3)})$

G-**Hom-associative** Hom-algebra (V, μ, α)

$$\sum_{s \in G \subseteq \mathcal{S}_3} (-1)^{\varepsilon(s)} \underbrace{(\mu(\mu(x_{s(1)}, x_{s(2)}), \alpha(x_{s(3)})) - \mu(\alpha(x_{s(1)}), \mu(x_{s(2)}, x_{s(3)})))}_{a_{\mu,\alpha}} =$$

$$\sum_{s \in G \subseteq \mathcal{S}_3} (-1)^{\varepsilon(s)} \underbrace{((x_{s(1)} x_{s(2)}) \alpha(x_{s(3)}) - \alpha(x_{s(1)})(x_{s(2)} x_{s(3)}))}_{a_{\alpha}} = 0,$$

$\Leftrightarrow$

$$\sum_{s \in G} (-1)^{\varepsilon(s)} a_{\mu,\alpha} \circ \sigma = 0$$

Hom-associator (α-associator): $a_{\alpha}(x, y, z) = (xy)\alpha(z) - \alpha(x)(yz)$

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

Theorem

G-Hom-associative algebras are Hom-Lie admissible:
For any G-Hom-associative algebra (V, μ, α),
$(V, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra
with commutator multiplication $[x, y] = \mu(x, y) - \mu(y, x)$

- **skew-symmetry** $\boxed{[x, y]_\alpha = -[y, x]_\alpha}$
- **Hom-Lie Jacobi identity** $\forall x, y, z \in L$

$$\sum_{\circlearrowleft_{(x,y,z)}} \big[\alpha(x), [y, z]_\alpha\big]_\alpha =$$

$$\boxed{\big[\alpha(x), [y, z]_\alpha\big]_\alpha + \big[\alpha(z), [x, y]_\alpha\big]_\alpha + \big[\alpha(y), [z, x]_\alpha\big]_\alpha = 0}$$

G-Hom-associative algebras

The subgroups of $\mathcal{S}_3$ are

$G_1 = \{Id\}, \quad G_2 = \{Id, \tau_{12}\}, \quad G_3 = \{Id, \tau_{23}\}$

$G_4 = \{Id, \tau_{13}\}, \quad G_5 = A_3, \quad G_6 = \mathcal{S}_3$

A_3 is the alternating group;

τ_{ij} is the transposition of i and j.

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

- The G_1-Hom-associative algebras are the Hom-associative algebras.
- The G_2-Hom-associative algebras satisfy

$$\mu(\alpha(x),\mu(y,z))-\mu(\alpha(y),\mu(x,z))=\mu(\mu(x,y),\alpha(z))-\mu(\mu(y,x),\alpha(z))$$

Vinberg algebra or left symmetric algebra: $\alpha = Id$

- The G_3-Hom-associative algebras satisfy

$$\mu(\alpha(x),\mu(y,z))-\mu(\alpha(x),\mu(z,y))=\mu(\mu(x,y),\alpha(z))-\mu(\mu(x,z),\alpha(y))$$

Hom-pre-Lie algebra

$$\boxed{\alpha(x)(yz) - \alpha(x)(zy) = (xy)\alpha(z) - (xz)\alpha(y)}$$

Pre-Lie algebra or right symmetric algebra: $\alpha = Id$

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

- The G_4-Hom-associative algebras satisfy

$$\mu(\alpha(x), \mu(y, z)) - \mu(\alpha(z), \mu(y, x)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(z, y), \alpha(x))$$

- The G_5-Hom-associative algebras satisfy the condition

$$\mu(\alpha(x), \mu(y, z)) + \mu(\alpha(y), \mu(z, x)) + \mu(\alpha(z), \mu(x, y)) = \mu(\mu(x, y), \alpha(z)) + \mu(\mu(y, z), \alpha(x)) + \mu(\mu(z, x), \alpha(y))$$

Note: μ skew-symmetric $\Rightarrow$ the Hom-Jacobi identity.

- The G_6-Hom-associative algebras are the Hom-Lie admissible algebras.

From Lie algebras to Hom-Lie algebras. Composition trick

$(V, [\cdot,\cdot])$ Lie algebra
$\alpha : V \to V$ Lie algebra endomorphism

$$[x,y]_\alpha = \alpha([x,y])$$

Then $(V, [\cdot,\cdot]_\alpha)$ is a Hom-Lie algebra

$$[x,y]_\alpha = -[y,x]_\alpha, \qquad \circlearrowleft_{x,y,z} [[\alpha(x),[y,z]_\alpha]_\alpha = 0.$$

Hom-Leibniz algebras (Hom-Loday algebras)

Special subclass of quasi-Leibniz algebras

Quasi-Leibniz algebras were first introduced in

D. Larsson, S. Silvestrov, Quasi-Lie algebras, In: Jurgen Fuchs, et al. (eds), "Noncommutative Geometry and Representation Theory in Mathematical Physics", American Mathematical Society, Contemporary Mathematics, Vol. 391, 2005.

Definition $(V, \langle\cdot,\cdot\rangle, \alpha)$ consisting of a linear space V, bilinear map $\langle\cdot,\cdot\rangle : V \times V \to V$ and a homomorphism $\alpha : V \to V$ satisfying

$$\langle\langle x, y\rangle, \alpha(z)\rangle = \langle\langle x, z\rangle, \alpha(y)\rangle + \langle\alpha(x), \langle y, z\rangle\rangle.$$

If a Hom-Leibniz algebra is skewsymmetric then it is a Hom-Lie algebra.

Hom-Poisson algebra

Makhlouf A., Silvestrov S.D., Notes on 1-Parameter Formal Deformations of Hom-associative and Hom-Lie Algebras, Forum Math. 22 (4), 715-739 (2010).

(Preprints in Mathematical Sciences, Lund University, Centre for Mathematical Sciences, Centrum Scientiarum Mathematicarum, (2007:31) LUTFMA-5095-2007 (2007); arXiv:0712.3130 (2007)).

Definition

$(V, \mu, \{\cdot,\cdot\}, \alpha)$, V linear space, $\mu : V \times V \to V$ and $\{\cdot,\cdot\} : V \times V \to V$ bilinear maps, $\alpha : V \to V$ linear map:

1) (V, μ, α) is a commutative Hom-associative algebra
2) $(V, \{\cdot,\cdot\}, \alpha)$ is a Hom-Lie algebra
3) for all x, y, z in V,

$$\{\alpha(x), \mu(y,z)\} = \mu(\alpha(y), \{x,z\}) + \mu(\alpha(z), \{x,y\}).$$

Equivalently, for all x, y, z in V, $ad_z(\cdot) = \{\cdot, z\}$ is a Hom-derivation for the multiplication μ:

$$\{\mu(x,y), \alpha(z)\} = \mu(\{x,z\}, \alpha(y)) + \mu(\alpha(x), \{y,z\})$$

Hom-Poisson algebra and deformations

Let $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ be a deformation of the commutative Hom-associative algebra
$\mathcal{A}_0 = (V, \mu_0, \alpha_0)$

$$\mu_t(x,y) = \mu_0(x,y) + \mu_1(x,y)t + \mu_2(x,y)t^2 + \dots$$

Then

$$\frac{\mu_t(x,y) - \mu_t(y,x)}{t} = \mu_1(x,y) - \mu_1(y,x) + t\sum_{i\geq 2}(\mu_i(x,y) - \mu_i(y,x))t^{i-2}$$

Formally,

$$\lim_{t\to 0} \frac{\mu_t(x,y) - \mu_t(y,x)}{t} = \{x,y\} := \mu_1(x,y) - \mu_1(y,x)$$

$(= xy - yx)$ commutator for μ_1

Hom-Poisson algebra

Theorem

$\mathcal{A}_0 = (V, \mu_0, \alpha_0)$
a commutative Hom-associative algebra
$\mathcal{A}_t = (V, \mu_t, \alpha_t)$ a deformation of $\mathcal{A}_0$.
Consider the bracket
$\{x, y\} = \mu_1(x, y) - \mu_1(y, x)$
is the first order element of the deformation μ_t.

$$\Downarrow$$

$(V, \mu_0, \{,\}, \alpha_0)$ is a Hom-Poisson algebra.

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Thank you for the music,

the songs I'm singing!

Thanks for all the joy

they're bringing!