Non-associative skew Laurent polynomial rings

The 5th meeting of the Swedish Network for Algebra and Geometry, Mälardalen University, 2023

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Mälardalen University

Outline

I. Background and motivation

II. Skew Laurent polynomial rings

III. Non-associative skew Laurent polynomial rings

IV. Hilbert's basis theorem

Background and motivation

Non-commutative rings with a *skewed* or *twisted* multiplication; *Hilbert's twist* [Hil03].

Appear as *universal enveloping algebras* of Lie algebras, *quantized coordinate rings* of affine algebraic varieties, *group rings*, *crossed products* etc. Used e.g. in coding theory.

Ore extensions were introduced by Ore in [Ore33], and *non-associative Ore extensions* in [NÖR18]. What about *non-associative skew Laurent polynomial rings*?

We generalize results on simplicity and Hilbert's basis theorem – with some surprises!

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Skew Laurent polynomial rings

Definition (Skew Laurent polynomial ring)

(S1) S is a free left R-module with basis $\{1, x, x^{-1}, x^2, x^{-2}, \ldots\}$.

(S2) $xR = Rx$.

(S3) S is associative.

Let R be an associative ring with an automorphism σ. The *generalized Laurent polynomial ring* $R[X^{\pm}; \sigma]$ is $\left\{\sum_{i \in \mathbb{Z}} r_i X^i : r_i \in R \text{ zero for all but finitely many } i \in \mathbb{Z}\right\}$. Addition is pointwise and multiplication defined by

$$(rX^m)(sX^n) = (r\sigma^m(s))\, X^{m+n}, \quad r, s \in R,\ m, n \in \mathbb{Z}.$$

The *generalized polynomial ring* $R[X; \sigma] \subset R[X^{\pm}; \sigma]$ subset of sums with $i \in \mathbb{N}$.

Proposition

$R[X^{\pm}; \sigma]$ is a skew Laurent polynomial ring of R with $x = X$.

Proposition

Every skew Laurent polynomial ring of R is isomorphic to a generalized Laurent polynomial ring $R[X^{\pm}; \sigma]$.

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What are examples of skew Laurent polynomial rings?

Example

Let R be an associative ring. Then $R[X^{\pm}] = R[X^{\pm}; \mathrm{id}_R]$ (and $R[X] = R[X; \mathrm{id}_R]$).

Example

Let $* : \mathbb{C} \to \mathbb{C}$, $u \mapsto u^*$ be complex conjugation. In $\mathbb{C}[X^{\pm}; *]$ (and $\mathbb{C}[X; *]$), $Xu = u^* X$.

We have $\mathbb{C} \cong \mathbb{R}[X]/(X^2 + 1)$ and $\mathbb{H} \cong \mathbb{C}[X; *]/(X^2 + 1)$.

Example

Let K be a field. The *quantum torus* $T_q(K)$ is $K\langle X^{\pm}, Y^{\pm}\rangle/(XY - qYX)$ for some $q \in K^{\times}$.
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Non-associative skew Laurent polynomial rings

Convention. A *non-associative ring* is a ring which is not necessarily associative.

Definition (Associator and nuclei)

(A, B, C) finite sums (a, b, c) with $a \in A$, $b \in B$, $c \in C$ for $A, B, C \subseteq R$.

$N_l(R) := \{r \in R \colon (r, s, t) = 0 \text{ for all } s, t \in R\}$. $N_m(R)$ and $N_r(R)$ defined similarly.

Definition (Left R-module)

If R is a non-associative ring, a *left R-module* is an additive group M with a biadditive map $R \times M \to M$, $(r, m) \mapsto rm$ for any $r \in R$ and $m \in M$ (if it has a basis, then it's *free*).

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Definition (Associator and nuclei)

$(\cdot,\cdot,\cdot)\colon R\times R\times R\to R$ is defined by $(r,s,t):=(rs)t-r(st)$ for $r,s,t\in R$.

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Definition (Left R-module)

If R is a non-associative ring, a *left R-module* is an additive group M with a biadditive map $R\times M\to M$, $(r,m)\mapsto rm$ for any $r\in R$ and $m\in M$ (if it has a basis, then it's *free*).

Convention. A *non-associative ring* is a ring which is not necessarily associative.

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Definition (Non-associative skew Laurent polynomial ring)

(N1) S is a free left R-module with basis $\{1, x, x^{-1}, x^2, x^{-2}, \ldots\}$.

(N2) $xR = Rx$.

(N3) $(S, S, x) = (S, x, S) = \{0\}$.

Let R be a **non-associative ring** with an **additive bijection σ that respects 1**.
The *generalized Laurent polynomial ring* $R[X^{\pm}; \sigma]$ is defined as in the associative case,

$$(rX^m)(sX^n) = (r\sigma^m(s))X^{m+n}, \quad r, s \in R, \ m, n \in \mathbb{Z}.$$

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Proposition ([BR22])

*$R[X^{\pm}; \sigma]$ is a **non-associative** skew Laurent polynomial ring of R with $x = X$.*

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Example ([BR22])

$R[X^{\pm}] = R[X^{\pm}; \mathrm{id}_R]$ is associative $\iff R$ is associative.

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On $\mathbb{C}$, define $\sigma(a + bi) = a + qbi$ for any $a, b \in \mathbb{R}$ and $q \in \mathbb{R}^{\times}$.
Then σ is an automorphism $\iff q = \pm 1 \iff \mathbb{C}[X^{\pm}; \sigma]$ is associative.

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The *quantum torus* $T_q(\mathbb{O})$ for any $q \in \mathbb{R}^{\times}$ is $\mathbb{O} \otimes_{\mathbb{R}} T_q(\mathbb{R})$. Then $T_q(\mathbb{O})$ is (isomorphic to) $\mathbb{O}[Y^{\pm}][X^{\pm}; \sigma]$ where σ is the $\mathbb{O}$-automorphism $\sigma(Y) = qY$. $T_q(\mathbb{O})$ is not associative.

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Let A be any of the real $*$-algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \ldots$ with $*$ conjugation.
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Non-associative skew Laurent polynomial rings

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Hilbert’s basis theorem

A family $\mathcal{F}$ of subsets satisfies the *ascending chain condition* if there is no infinite chain $S_1 \subset S_2 \subset \dots$ and $S_1, S_2, \dots \in \mathcal{F}$. $S \in \mathcal{F}$ is *maximal* if no $T \in \mathcal{F}$ with $S \subset T$.

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(NR1) *R satisfies the ascending chain condition on its right (left) ideals.*

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Theorem (Hilbert's basis theorem)

Let R be an associative, commutative ring. If R is Noetherian, then so is $R[X]$.

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Remark. If R is left Noetherian, then $R[X;\sigma]$ is *not* necessarily left Noetherian!

Q: Is there a (non-)associative, left Noetherian ring R where $R[X^{\pm};\sigma]$ is *not* left Noetherian? ($R[X^{\pm};\sigma]$ is naturally *strongly graded* by $\mathbb{Z}$ with principal component R.)

Corollary ([BR22])

Let D be a non-associative division ring with an additive bijection σ that respects 1.
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Can consider *skew power series rings* $R[[X;\sigma]]$ as $\left\{\sum_{i=0}^{\infty} r_i X^i : r_i \in R\right\}$ and *skew Laurent series rings* $R((X;\sigma))$ as $\left\{\sum_{i=k}^{\infty} r_i X^i : r_i \in R, k \in \mathbb{Z}\right\}$ with $(rX^m)(sX^n) = (r\sigma^m(s))X^{m+n}$.

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