Other constructions of hom-associative algebras

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Abstract

A hom-algebra is an algebra that in addition to ordinary multiplication μ also has a linear unary operation α, known as the hom or twisting map. There exist a number of hom-variants of ordinary classes of algebras, where axioms have been modified by inserting homs in select positions: for example, hom-Lie and hom-associative algebras satisfy homified variants of the Jacobi and associativity respectively axioms.
A potentially troubling matter is however, that examples of hom-associative algebras found in the literature are pretty much exclusively Yau twists of associative algebras, which raises the worry that they might all be just ordinary associative algebras presented in a silly way. This talk puts that worry to rest by presenting two new constructions—the homassocification of a general algebra, and hom-associative algebras with truncated freedom—that can produce non-Yau-twist hom-associative algebras. The latter construction also provides the first explicit example of a hom-associative algebra that is not strongly hom-associative.

Hom-algebra

A hom-algebra is a triple (H, μ, α) where H is a vector space (or module), $\mu \colon H \times H \longrightarrow H$ is a bilinear map (the *multiplication*), and $\alpha \colon H \longrightarrow H$ is a linear map (the *hom* or *twist*).
(H, μ, α) is hom-associative if

$$\mu\bigl(\mu(a,b), \alpha(c)\bigr) = \mu\bigl(\alpha(a), \mu(b,c)\bigr) \qquad \text{for all } a, b, c \in H.$$

This may be viewed as a *homogenised* version of ordinary associativity (in that every variable goes through the same number of operations in the left hand side as it does in the right hand side), but that is not the etymology of 'hom'.

Hom-associativity example

The standard construction of hom-associative algebras is to do a Yau twist:

Pick an *associative* algebra H and an endomorphism α of that algebra. Define

$$\mu(a,b) = \alpha(a \cdot b) \qquad \text{for all } a,b \in H.$$

Now (H,μ,α) is a hom-associative algebra.

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If you look carefully, there probably is *some* non-Yau-twist example out there, but it doesn't seem to be much publicised.

For the research area, this could be a risky state of affairs, because it means we don't have a good answer to:

> *How do we know you're not just studying associative algebras under a contorted presentation?*

Well, here follows some other examples.

Hom-assocification

Definition

Let an *associative* algebra A, a (non-associative) algebra B, and an algebra homomorphism $f\colon B \longrightarrow A$ be given. Then the hom-assocification of B (or of the diagram $B \xrightarrow{f} A$) is $(B \times A, \mu, \alpha)$, where

$$\mu\bigl((b_1, a_1), (b_2, a_2)\bigr) = \bigl(b_1 b_2, f(b_1)a_2 + a_1 a_2 + a_1 f(b_2)\bigr),$$
$$\alpha(b_1, a_1) = \bigl(0, f(b_1) + a_1\bigr)$$

for all $b_1, b_2 \in B$ and $a_1, a_2 \in A$.

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This looks a bit like the construction of adding a unit to a non-unital associative R-algebra A by defining

$$\mu\bigl((r_1,a_1),(r_2,a_2)\bigr) = (r_1 r_2, r_1 a_2 + a_1 a_2 + r_2 a_1) \quad \text{for all } (r_1,a_1),(r_2,a_2) \in R \times A.$$

But there R is typically smaller than A, whereas B is typically larger than A.

Motivation

Back in 2009, Frégier & Gohr observed that the standard unitality axiom ($\mu(1,x) = x = \mu(x,1)$ for all x) interacts poorly with hom-associativity: it implies

$$\mu\bigl(\alpha(x),y\bigr) = \mu\bigl(x,\alpha(y)\bigr) = \alpha\bigl(\mu(x,y)\bigr) \qquad \text{for all } x,y \in H.$$

Such an α cannot sense much of its argument, if applying it to x or to y has the same overall effect!

(From the hom side of things, the problem is that the unitality axiom is not homogeneous. If hom-ified to

$$\mu(1,x) = \alpha(x) = \mu(x,1)$$

then you can freely add such a hom-unit 1 to your algebra again. But that is not our angle here.)

Effect on monomials

Being able to move α to any position around a μ means that in a monomial you can move that α to *any position you like*. This means there are two kinds of monomials:

- monomials not containing α, that are completely unconstrained by hom-associativity, and
- monomials containing at least one α, that are effectively associative since you can always move that α to where the hom-associativity axiom would need it to be

in any hom-associative algebra (H, μ, α) which is unital in the *standard sense* (rather than hom-unital).

This ought to carry over to a monomial basis of H...

One might expect such an algebra (H, μ, α) to decompose as

$$H = B \oplus A$$

where A is an associative subalgebra, B is just a subalgebra, and $\alpha\colon H \longrightarrow A$.

I don't know whether that is always the case, but hom-assocification goes the *opposite* way: start with B and associative A, then glue them together as a hom-associative algebra.

In particular B and A are both embedded into the hom-associficaiton.

If you as B pick something definitely not a Yau twist, then you get a hom-associative algebra that is not a Yau twist.

Exhibiting examples

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- Jordan algebras? That works, but then you get a *commutative* hom-algebra, so the corresponding hom-Lie algebra is abelian.

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- Jordan algebras? That works, but then you get a *commutative* hom-algebra, so the corresponding hom-Lie algebra is abelian.
- Ore extensions? (says Per Bäck). Now we're talking!
  It turns out it's easy to make some Ore extension $R[x;\sigma,\delta]$ an algebra, but requires strict constraints on σ and δ to make it associative.

Ore extensions

Definition

Let R be an algebra (not necessarily unital) and $\sigma, \delta \colon R \longrightarrow R$ linear maps. Then the Ore extension $R[x;\sigma,\delta]$ is $\bigoplus_{n\in\mathbb{N}} Rx^n$ ($|\mathbb{N}|$ copies of R, notationally distinguished by power of x), with multiplication $\cdot$ recursively defined by

$$rx^0 \cdot sx^n = (rs)x^n,$$
$$rx^{m+1} \cdot sx^n = rx^m \cdot \bigl(\sigma(s)x^{1+n} + \delta(s)x^n\bigr)$$

for all $r,s \in R$ and $m,n \in \mathbb{N}$.

Lemma

$R[x;\sigma,\delta]$ as above is an algebra.
If R is associative, σ is an algebra homomorphism, and δ is a σ-derivation then $R[x;\sigma,\delta]$ is associative.

How to build an example

Lemma

Let $R_1[x;\sigma_1,\delta_1]$ and $R_2[x;\sigma_2,\delta_2]$ be two Ore extensions. If an algebra homomorphism $\varphi\colon R_1 \longrightarrow R_2$ satisfies $\sigma_2 \circ \varphi = \varphi \circ \sigma_1$ and $\delta_2 \circ \varphi = \varphi \circ \delta_1$ then the linear map $f\colon R_1[x;\sigma_1,\delta_1] \longrightarrow R_2[x;\sigma_2,\delta_2]$ defined by

$$f(rx^n) = \varphi(r)x^n \qquad \text{for all } r \in R_1 \text{ and } n \in \mathbb{N}$$

is an algebra homomorphism.

Plan: Do something naughty causing $B = R_1[x;\sigma_1,\delta_1]$ to be nonassociative, but which is killed off by φ, so that $A = R_2[x;\sigma_2,\delta_2]$ becomes associative. Then hom-assocify $R_1[x;\sigma_1,\delta_1] \xrightarrow{f} R_2[x;\sigma_2,\delta_2]$.

Keeping it simple

Take $R_2 = \mathbb{C}[y]$, $\sigma_2(r) = r$, and $\delta_2 = \frac{d}{dy}$.
Then $A := R_2[x; \sigma_2, \delta_2]$ is the plain old Heisenberg–Weyl algebra.

Pick $R_1 = \mathbb{C}[y_1, y_2]$ with $\varphi(y_1) = y$ and $\varphi(y_2) = y^2$, because why not?
Can take δ_1 as the derivation with $\delta_1(y_1) = 1$ and $\delta_1(y_2) = 2y_1$.

Finally, to make the $\varphi = \varphi \circ \sigma_1$, we can take

$$\sigma_1(r) = r + dg(r) \qquad \text{for } r \in R_1$$

where $d \in \ker \varphi$ and $g\colon R_1 \longrightarrow R_1$ is some arbitrary linear map.

Easiest is $d = y_2 - y_1^2$ and $g(r) = r$—can work out a closed form formula for multiplication in $B := R_1[x; \sigma_1, \delta_1]$ in this case. (Easy because σ_1 and δ_1 commutes.)
More interesting might be $g(y_1^i y_2^k) = y_1^k y_2^i$.

Backing off

In a sense, the hom-assocification works by having one *low part* where things can be complicated, and a *high part* which collapses into some simpler, familiar structure.

Hom-assocification draws the line between having no α or some α, making the high part associative and the low part whatever, but one doesn't have to draw the line there.

One could take a fragment of the free hom-associative algebra, defined via explicit multiplication table, as low part. Any monomial product not contained in this fragment ends up in the simplified high part.
This is the idea for truncated freedom hom-associative algebras.

Small example

If we consider a hom-associative algebra generated by one element x, and put the limit at one α and two μ in our low part monomials, then we can denote the low basis elements as e_{nmk}, where

$$\begin{aligned} n &= \text{power of } x, \\ m &= \text{times } \alpha \text{ applied} = \text{power of } y, \\ k &= \text{extra index for distinction.} \end{aligned}$$

If e_{nmk} is part of a product that ends up in the high part, it behaves as the commutative polynomial $x^n y^m$.

$$\begin{aligned} e_{100} &= x & e_{211} &= \mu(x, \alpha(x)) & e_{310} &= \mu(e_{110}, e_{200}) & e_{315} &= \mu(e_{210}, x) \\ e_{110} &= \alpha(x) & e_{212} &= \alpha(\mu(x, x)) & e_{311} &= \mu(x, e_{210}) & e_{316} &= \mu(e_{211}, x) \\ e_{200} &= \mu(x, x) & e_{300} &= \mu(x, e_{200}) & e_{312} &= \mu(x, e_{211}) & e_{310} &= \mu(e_{200}, e_{110}) \\ e_{210} &= \mu(\alpha(x), x) & e_{301} &= \mu(e_{200}, x) & e_{313} &= \mu(x, e_{212}) & e_{317} &= \mu(e_{212}, x) \\ & & & & e_{314} &= \alpha(e_{300}) & e_{318} &= \alpha(e_{301}) \end{aligned}$$

The table starts

μ	e_{100}	e_{110}	e_{200}	e_{210}	e_{211}	e_{212}	e_{300}	e_{301}
e_{100}	e_{200}	e_{211}	e_{300}	e_{311}	e_{312}	e_{313}	x^4	x^4
e_{110}	e_{210}	x^2y^2	e_{310}	x^3y^2	x^3y^2	x^3y^2	x^4y	x^4y
e_{200}	e_{301}	e_{310}	x^4	x^4y	x^4y	x^4y	x^5	x^5
e_{210}	e_{315}	x^3y^2	x^4y	x^4y^2	x^4y^2	x^4y^2	x^5y	x^5y
e_{211}	e_{316}	x^3y^2	x^4y	x^4y^2	x^4y^2	x^4y^2	x^5y	x^5y
e_{212}	e_{317}	x^3y^2	x^4y	x^4y^2	x^4y^2	x^4y^2	x^5y	x^5y
e_{300}	x^4	x^4y	x^5	x^5y	x^5y	x^5y	x^6	x^6
e_{301}	x^4	x^4y	x^5	x^5y	x^5y	x^5y	x^6	x^6

$$\alpha(e_{100}) = e_{110}$$
$$\alpha(e_{110}) = xy^2$$
$$\alpha(e_{200}) = e_{212}$$
$$\alpha(e_{210}) = x^2y^2$$
$$\alpha(e_{211}) = x^2y^2$$
$$\alpha(e_{212}) = x^2y^2$$
$$\alpha(e_{300}) = e_{314}$$
$$\alpha(e_{301}) = e_{317}$$

Practically one may want to restrict to a smaller set of monomials as basis for the low part: only those which are needed to reach certain monomials.

Why do this?

An earlier systematic study of the consequences of the hom-associativity axiom has revealed that there is a large class of equalities—the canyon identities—that *almost* hold in all hom-associative algebras, but not quite.

Favourite example:

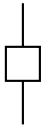
$$\mu\Big(\mu(x,\alpha(x)),\mu(\mu(x,x),x)\Big) \neq \mu\Big(\mu(x,\mu(x,x)),\mu(\alpha(x),x)\Big)$$

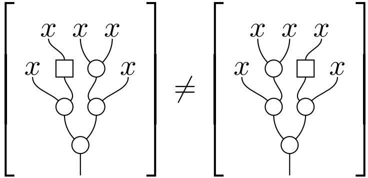
in the free hom-associative algebra, but

$$\mu\Big(\mu\Big(\mu(x,\alpha(x)),\mu(\mu(x,x),x)\Big),\alpha(\alpha(x))\Big) =$$
$$= \mu\Big(\mu\Big(\mu(x,\mu(x,x)),\mu(\alpha(x),x)\Big),\alpha(\alpha(x))\Big)$$

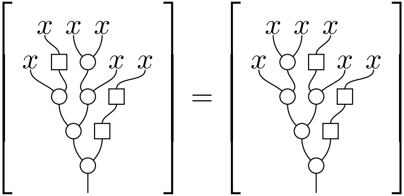
so the free algebra has zero-divisors.

Again, with graphs

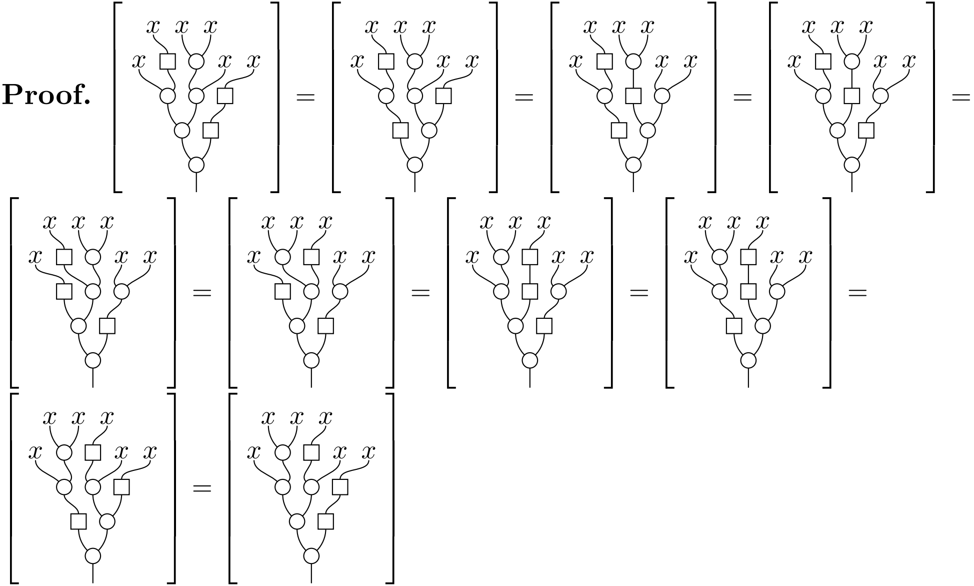
Writing  for μ and  for α, we have



in the free hom-associative algebra, but



Proof.



Strong and weak hom-associativity

I have previously defined the class of strongly hom-associative algebras, where all the canyon identities hold.

- Have confluent rewrite theory, and thus combinatorial description of the basis of the free algebra.
- All Yau twists and hom-associfications are strongly hom-associative.

However, one would like to have an explicit example of a hom-associative algebra that is merely weakly hom-associative, in that it violates some canyon identity.
AFAIK, the truncated freedom hom-associative algebras provide the first such explicit examples.

That's all, folks!