Symmetric (σ, τ)-derivations and (σ, τ)-Hochschild cohomology

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Introduction

Definition

Let σ and τ be endomorphisms of an associative algebra $\mathcal{A}$ and let M be a $\mathcal{A}$-bimodule. A $\mathbb{K}$-linear map $X : \mathcal{A} \to M$ is called a (σ, τ)-derivation with values in M if

$$X(fg) = \sigma(f)X(g) + X(f)\tau(g) \tag{3.1}$$

for all $f, g \in \mathcal{A}$.

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The Jackson derivative D_q on the polynomial algebra $\mathbb{C}[x]$. The Jackson derivative is defined as

$$D_q(f(x)) = \frac{f(qx) - f(x)}{(q-1)x} \tag{3.2}$$

for $f(x) \in \mathbb{C}[x]$ and $q \in \mathbb{C}, q \neq 1$.

Definition

A (σ, τ)-derivation X with values in M is called *inner* if there exists $m_0 \in M$ such that

$$X(f) = m_0 \tau(f) - \sigma(f) m_0$$

for all $f \in \mathcal{A}$.

- We are interested in describing (σ, τ)-derivations using Hochschild cohomology.
- We also interested in the following question: Is the Jackson derivative an outer (σ, τ)-derivation?

Hochschild cohomology

Denote

$$\mathcal{A}^{\otimes n} = \mathcal{A} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{A}$$

the n-fold tensor product,

$$C^n(\mathcal{A}, M) = \{\omega : \mathcal{A}^{\otimes n} \to M\},$$

for $n \geq 1$ linear spaces of $\mathbb{C}$-linear space consisting of cochains and

$$C^0(\mathcal{A}, M) = M.$$

Consider a sequence of linear spaces of cochains

$$0 \to M \to C^1(\mathcal{A}, M) \to \cdots \to C^n(\mathcal{A}, M) \to C^{n+1}(\mathcal{A}, M) \to \cdots$$

with differential

$$\delta_n : C^n(\mathcal{A}, M) \to C^{n+1}(\mathcal{A}, M).$$

Hochschild cohomology

The Hochschild cochain complex $C^*(\mathcal{A}, M, \delta)$ is the sequence of linear spaces of cochains together with the differential δ_n as the boundary map defined as

$$(\delta_0 m)(a) = ma - am$$

for $m \in M$ and $a \in A$ and

$$\begin{aligned} (\delta_n \omega)(a_1, \ldots, a_{n+1}) &= a_1 \omega(a_2, \ldots, a_{n+1}) \\ &+ \sum_{j=1}^{n} (-1)^j \omega(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) \\ &+ (-1)^{n+1} \omega(a_1, \ldots, a_n) a_{n+1} \end{aligned}$$

for $n \geq 1$.
The differential δ_n satisfies

$$\delta_n \circ \delta_{n-1} = 0.$$

Hochschild cohomology

Denote the set of n-cocycles of $\mathcal{A}$ for M,

$$Z^n(\mathcal{A}, M) = \{\omega \in C^n(\mathcal{A}, M) : \delta_n \omega = 0\},$$

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the set of n-coboundaries of $\mathcal{A}$ for M,

$$B^n(\mathcal{A}, M) = \{\omega \in C^n(\mathcal{A}, M) : \omega = \delta_{n-1}\rho, \ \text{ for } \rho \in C^{n-1}(\mathcal{A}, M)\}.$$

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The n-dimensional cohomology group of the algebra $\mathcal{A}$ for the bimodule M denoted $H^n(\mathcal{A}, M)$ is given by

$$H^n(\mathcal{A}, M) = Z^n(\mathcal{A}, M)/B^n(\mathcal{A}, M).$$

(σ, τ)-Hochschild cohomology

Let $\sigma, \tau \in \mathrm{End}(\mathcal{A})$ and define

$$f \cdot m = \sigma(f)m, \quad m \cdot f = m\tau(f).$$

Denote the new bimodule $M_{(\sigma,\tau)}$ and consider the Hochschild cohomology cochain complex $C^*(\mathcal{A}, M_{(\sigma,\tau)}, \delta)$.
The boundary map is given by

$$(\delta_0 m)(a) = m \cdot a - a \cdot m = m\tau(a) - \sigma(a)m$$

and

$$\begin{aligned}
&(\delta_n\omega)(a_1, \dots, a_{n+1}) = a_1 \cdot \omega(a_2, \dots, a_{n+1}) \\
&+ \sum_{j=1}^{n} (-1)^j \omega(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1}) \\
&+ (-1)^{n+1} \omega(a_1, \dots, a_n) \cdot a_{n+1} \\
&= \sigma(a_1)\omega(a_2, \dots, a_{n+1}) + \sum_{j=1}^{n} (-1)^j \omega(a_2, \dots, a_j a_{j+1}, \dots, a_{n+1}) \\
&+ (-1)^{n+1} \omega(a_1, \dots, a_n)\tau(a_{n+1}).
\end{aligned}$$

(σ, τ)-Hochschild cohomology

$$H^0(\mathcal{A}, M_{(\sigma,\tau)}) = \{m \in M_{(\sigma,\tau)} : m\tau(a) = \sigma(a)m, \text{ for } a \in \mathcal{A}\}.$$

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1-cocycles are (σ, τ)-derivations, i.e.,

$$\omega(a_1 a_2) = \sigma(a_1)\omega(a_2) + \omega(a_1)\tau(a_2)$$

and 1-coboundaries are inner (σ, τ)-derivations, i.e.,

$$(\delta_0 m)(a) = m\tau(a) - \sigma(a)m.$$

The degree 1 Hochschild cohomology group describes the outer (σ, τ)-derivations of $\mathcal{A}$, i.e.,

$$H^1(\mathcal{A}, M_{(\sigma,\tau)}) \quad = \quad \mathrm{Der}_{(\sigma,\tau)}(\mathcal{A}, M)/\mathrm{Inn}_{(\sigma,\tau)}(\mathcal{A}, M)$$

Symmetric (σ, τ)-derivations

Definition

A (σ, τ)-derivation with values in M is called *symmetric* if it is also a (τ, σ)-derivation with values in M.

Proposition

Let σ, τ be endomorphisms of $\mathcal{A}$ and let M be a $\mathcal{A}$-bimodule. If $m_0 \in M$ such that $[m_0, \tau(f)] = [m_0, \sigma(f)] = 0$ for all $f \in \mathcal{A}$ then $X : \mathcal{A} \to M$ given by

$$X(f) = m_0 \tau(f) - \sigma(f) m_0$$

is a symmetric inner (σ, τ)-derivation with values in M.

Regular condition

Definition

Let σ, τ be endomorphisms of $\mathcal{A}$. The pair (σ, τ) is called *regular* if there exists $f \in \mathcal{A}$ such that $\delta(f) = \tau(f) - \sigma(f)$ is not a zero divisor. Moreover, the pair (σ, τ) is called *strongly regular* if there exists $f \in \mathcal{A}$ such that $\delta(f)$ is invertible.

Lemma

If X is a symmetric (σ, τ)-derivation with values in a $\mathcal{A}$-bimodule M, then

$$X(f)\delta(g) = \delta(f)X(g) \tag{3.3}$$

for all $f, g \in \mathcal{A}$, where $\delta = \tau - \sigma$.

Proposition

Let (σ, τ) be a strongly regular pair of endomorphisms of $\mathcal{A}$ and let X be a symmetric (σ, τ)-derivation with values in a $\mathcal{A}$-bimodule M. Then there exists a unique $m_0 \in M$ such that

$$X(f) = m_0 \tau(f) - \sigma(f) m_0$$

and

$$[m_0, \tau(f)] = [m_0, \sigma(f)] = 0$$

for all $f \in \mathcal{A}$.

Corollary

If (σ, τ) is a strongly regular pair of endomorphisms of $\mathcal{A}$ then every symmetric (σ, τ)-derivation with values in a $\mathcal{A}$-bimodule M is inner.

(σ, τ)-Hochschild cohomology for regular pairs

Proposition

Let $\mathcal{A}$ be a commutative algebra and (σ, τ) be a regular pair. Then

$$H^0_{(\sigma,\tau)}(\mathcal{A}, \mathcal{A}) = 0.$$

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Proposition

Let σ and τ be endomorphisms of an associative algebra $\mathcal{A}$ and assume that every (σ, τ)-derivation on $\mathcal{A}$ with values in M be symmetric. If (σ, τ) is a strongly regular pair then

$$H^1_{(\sigma,\tau)}(\mathcal{A}, M) = 0 = H^1_{(\tau,\sigma)}(\mathcal{A}, M).$$

(σ, τ)-Hochschild cohomology for $\mathbb{K}[x]$

Proposition

Let $\mathbb{K}[x]$ *be the polynomial algebra over a field* $\mathbb{K}$ *and let*

$$I_\delta = \langle p(x)\delta(x) | p(x) \in \mathbb{K}[x]\rangle$$

be an ideal generated by $\delta(x)$*. Then*

$$H^1_{(\sigma,\tau)}(\mathbb{K}[x], \mathbb{K}[x]) = \mathbb{K}[x]/I_\delta.$$

Proposition

Let $\sigma, \tau \in End(\mathbb{K}[x])$ *given by* $\sigma(x) = qx$ *and* $\tau(x) = x$ *for* $q \neq 0, 1$*. Then*

$$H^1_{(\sigma,\tau)}(\mathbb{K}[x], \mathbb{K}[x]) = \{\lambda \mathbb{1} : \lambda \in \mathbb{K}\}.$$

Thank You Very Much For Your Attention