Group-Graded Rings with UGN

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March 28, 2023

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What conditions on R and G will guarantee that R has UGN?

One obvious answer: If R is the group ring of G over R_1, then R must have UGN.

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(**Example:** strongly graded ring.)

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Supramenability

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Local finiteness

Amenable group; free and full grading

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2. *Every ring freely and fully graded by G has UGN if and only if its base ring has UGN.*
3. *Every crossed product of G over a ring R_1 has UGN if and only if R_1 has UGN.*
4. *Every skew group ring of G over a ring R_1 has UGN if and only if R_1 has UGN.*

Supramenable group; free grading

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Theorem B

Let R be a ring that is freely graded by a supramenable group G. Then R has UGN if and only if R_1 has UGN.

Supramenable group; free grading

Theorem B

Let R be a ring that is freely graded by a supramenable group G. Then R has UGN if and only if R_1 has UGN.

Open Question

Let R be a ring that is freely graded by an amenable group G. If R_1 has UGN, does it necessarily follow that R has UGN?

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Theorem D

Any ring projectively graded by a locally finite group has UGN if and only if its base ring has UGN.

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Let G be a group and R a ring freely graded by G. Let $X \subseteq G$ be the support of R. If X is amenable with respect to G, then R has UGN if and only if R_1 has UGN.

Basis for Theorems A and B

Proposition A

Let G be a group and R a ring freely graded by G. Let $X \subseteq G$ be the support of R. If X is amenable with respect to G, then R has UGN if and only if R_1 has UGN.

This is proved by embedding R in the *translation ring* $T_G(X, T(\mathbb{Z}, R_1))$.

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Proposition B

Let G be a group and X a subset of G. Then the following two statements are equivalent.

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Proposition B

Let G be a group and X a subset of G. Then the following two statements are equivalent.

1. *For every ring R, the ring $T_G(X, R)$ has UGN if and only if R has UGN.*

Basis for Theorems A and B

Proposition A

Let G be a group and R a ring freely graded by G. Let $X \subseteq G$ be the support of R. If X is amenable with respect to G, then R has UGN if and only if R_1 has UGN.

This is proved by embedding R in the *translation ring* $T_G(X, T(\mathbb{Z}, R_1))$. Then we apply:

Proposition B

Let G be a group and X a subset of G. Then the following two statements are equivalent.

1. *For every ring R, the ring $T_G(X, R)$ has UGN if and only if R has UGN.*
2. *The subset X is amenable with respect to G.*

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For any group G, the following two assertions are equivalent.

1. (i) *G is amenable.*
2. (ii) *For any ring R, the ring $T(G, R)$ has UGN if and only if R has UGN.*

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Corollary

For any group G, the following two assertions are equivalent.

1. *G is amenable.*
2. *For any ring R, the ring $T(G, R)$ has UGN if and only if R has UGN.*

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For any group G, the following two assertions are equivalent.

1. *G is amenable.*
2. *For any ring R, the ring $T(G, R)$ has UGN if and only if R has UGN.*

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For any group G, the following two assertions are equivalent.

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For any group G, the following two assertions are equivalent.

1. *G is amenable.*
2. *For any ring R, the ring $T(G, R)$ has UGN if and only if R has UGN.*

Corollary

For any group G, the following two assertions are equivalent.

1. *G is supramenable.*
2. *For any ring R and nonempty set $X \subseteq G$, the ring $T_G(X, R)$ has UGN if and only if R has UGN.*