Lie algebra modules which are free over a subalgebra

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Part I

Lie algebras and their representations

Lie Algebras

Example

$\mathfrak{gl}_n = M_{n\times n}(\mathbb{C})$ with $[A, B] := AB - BA$ for $A, B \in \mathfrak{gl}_n$.

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Example

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\mathfrak{sl}_2 = \mathrm{span}(x, y, h)$, where $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$.

Universal enveloping algebra

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if $x_1, \ldots, x_n$ is an ordered basis for $\mathfrak{g}$, the monomials $\{x_1^{a_1} \cdots x_n^{a_n} | a_i \geq 0\}$ is a basis for $U(\mathfrak{g})$.

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Example

$$U(\mathfrak{sl}_2) = \mathrm{span}\{y^a h^b x^c \mid a, b, c \geq 0\}$$

Representation theory

A representation of $\mathfrak{g}$: a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}(V)$.

Equivalently, V is a Lie algebra module for $\mathfrak{g}$.

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Classification of simple modules

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But the problem is hard in general, J. Dixmier writes:

> But a deeper study reveals the existence of an enormous number of irreducible representations of $\mathfrak{h}$ [...]. It seems that these representations defy classification. A similar phenomenon exists for $\mathfrak{g} = \mathfrak{sl}_2$, and most certainly for all nonabelian Lie algebras.

Block's classification of simple $\mathfrak{sl}_2$ modules

Block's Theorem

Simple $\mathfrak{sl}_2$-modules come in three types:

- **Highest weight modules** - modules for which x has an eigenvector with eigenvalue zero, $x \cdot v = 0$.
- **Whittaker modules** - modules for which x has an eigenvector with nonzero eigenvalue, $x \cdot v = \lambda v$, $\lambda \neq 0$.
- **Third type modules** - these are in bijective correspondence with pairs $(\gamma, [a])$, where $\gamma \in \mathbb{C}$ and $[a]$ is a similarity class of irreducible elements of $\mathbb{C}(z)[\frac{d}{dx}]$.

Classification beyond $\mathfrak{sl}_2$ is hopeless

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Idea: study certain families of $\mathfrak{g}$-modules with "nice" properties such as

- Finite dimensional modules (Killing-Cartan 1913)
- Weight modules with finite dimensional weight spaces (Mathieu 2000)
- Modules in category $\mathcal{O}$
- Whittaker modules
- Gelfand-Zetlin modules

Triangular decomposition

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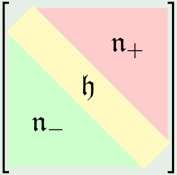

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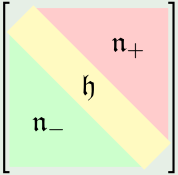
$\mathfrak{sl}_n =$ 

Decomposition of $\mathfrak{sl}_n$

For $\mathfrak{sl}_2$, we defined $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Take $\mathfrak{n}_+, \mathfrak{h}, \mathfrak{n}_-$ as the span of x, h, y respectively.

Weight modules

In a **weight module** V, the subalgebra $\mathfrak{h}$ acts diagonally:

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \qquad h \cdot v = \lambda(h) v \quad \text{for } h \in \mathfrak{h}, v \in V_\lambda$$

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Visualization of a weight module for $\mathfrak{sl}_2$

$$\begin{tikzcd}
\boxed{-4} \arrow[loop above, "h"] \arrow[r, bend left, "x"] & \boxed{-2} \arrow[loop above, "h"] \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] & \boxed{0} \arrow[loop above, "h"] \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] & \boxed{2} \arrow[loop above, "h"] \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] & \boxed{4} \arrow[loop above, "h"] \arrow[l, bend left, "y"]
\end{tikzcd}$$

More generally, in a **generalized weight module**, $\mathfrak{h}$ acts locally finitely: $\dim U(\mathfrak{h})v < \infty$ for each $v \in V$.

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Pick a central character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ and let I^χ be the ideal of $Z(\mathfrak{g})$ generated by elements $z - \chi(z)$. Kostant proved that the Whittaker module

$$W_\eta^\chi = W_\eta / I^\chi W_\eta$$

is simple and has central character χ.

Part II

Polynomial modules

Polynomial modules

Let $\mathfrak{a} \subset \mathfrak{g}$ be an abelian subalgebra.
Idea: study the category of $\mathfrak{g}$-modules where $\mathfrak{a}$ acts *freely* instead of locally finitely.

$U(\mathfrak{a})$ is a free $U(\mathfrak{a})$-module of rank 1, and if $x_1, \ldots, x_n$ is a basis for $\mathfrak{a}$, $U(\mathfrak{a}) \simeq K[x_1, \ldots, x_n]$.

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We define a corresponding category of $\mathfrak{a}$-free modules:

$$\mathcal{M}_{\mathfrak{a}}^{\mathfrak{g}} = \{V \in U(\mathfrak{g})\text{-Mod} \mid \mathrm{Res}_{U(\mathfrak{a})}^{U(\mathfrak{g})} V \text{ is free}\}$$

Taking $\mathfrak{a}$ as a Cartan subalgebra

The simple objects of rank 1 in $\mathcal{M}_{\mathfrak{h}}^{\mathfrak{g}}$ were classified 2015-2016.
Notably, $\mathcal{M}_{\mathfrak{h}}^{\mathfrak{g}}$ is empty when $\mathfrak{g}$ is not of type A or C.

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Example

We take $\mathfrak{a} = \mathfrak{h}$ the standard Cartan subalgebra for $\mathfrak{sl}_2$. For each $c \in \mathbb{C}$, let $M_c = \mathbb{C}[h]$ as a vector space and define

$$\begin{aligned} h \cdot f(h) &= hf(h), \\ x \cdot f(h) &= f(h-2), \\ y \cdot f(h) &= -\tfrac{1}{8}(h+c+2)(h-c)f(h+2). \end{aligned}$$

Then $M_c \in \mathcal{M}$. In Block's classification, this module belong to the Whittaker-class.

Generalizations

Various authors found corresponding classifications of $U(\mathfrak{h})$-free modules for

- Virasoro algebras
- Conformal algebras
- The Witt algebra
- Algebras of differential operators
- Heisenberg-Virasoro algebras
- Quantum algebras
- Super Lie algebras
- Kac-Moody algebras

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The simple objects of rank 1 in $\mathcal{M}_{\mathfrak{a}}^{\mathfrak{sl}_n}$ were classified in 2023.

Result

Simple objects of $\mathcal{M}_{\mathfrak{a}}^{\mathfrak{sl}_n}$ are parametrized by polynomials in $n-1$ variables.

How it looks for $\mathfrak{sl}_2$

Example

We take $\mathfrak{a} = \mathrm{span}(x)$. Then $U(\mathfrak{a}) \simeq k[x]$ as an $\mathfrak{a}$-module. Pick a polynomial $p \in k[x]$ and let $q := -\frac{1}{2x} \int_0^x p(t)p'(t) + tp''(t)dt$. Define a corresponding $\mathfrak{sl}_2$ action on $V(p) = k[x]$ as follows:

$$\begin{aligned} x \cdot f(x) &= xf(x), \\ h \cdot f(x) &= p(x)f(x) + 2xf'(x), \\ y \cdot f(x) &= q(x)f(x) - p(x)f'(x) - xf''(x). \end{aligned}$$

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Then $V(p) = \mathcal{M}_{\mathfrak{a}}^{\mathfrak{sl}_2}$. Moreover, any rank 1 module of $\mathcal{M}_{\mathfrak{a}}^{\mathfrak{sl}_2}$ is isomorphic to some $V(p)$.

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Then $V(p) = \mathcal{M}_{\mathfrak{a}}^{\mathfrak{sl}_2}$. Moreover, any rank 1 module of $\mathcal{M}_{\mathfrak{a}}^{\mathfrak{sl}_2}$ is isomorphic to some $V(p)$. These modules constitute a large family of modules of third Block-type.

Open questions

- For which pairs $(\mathfrak{a}, \mathfrak{g})$ is the category $\mathcal{M}_{\mathfrak{a}}^{\mathfrak{g}}$ nonempty?
- Does there exist simple objects in $\mathcal{M}_{\mathfrak{a}}^{\mathfrak{g}}$ of rank higher than one?
- Is there a correspondence between modules where $\mathfrak{a}$ acts freely and modules where $\mathfrak{a}$ acts locally nilpotently?
- Clebsch-Gordan problem: what can be said about the decomposition of $M \otimes E$ where $M \in \mathcal{M}_{\mathfrak{a}}^{\mathfrak{g}}$ and E is finite dimensional?
- What can be said about the category of modules which are finitely generated over $U(\mathfrak{a})$?

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