Kronecker algebras – a case study

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- I've mostly worked in a "derviation-based" approach to noncommutative geometry.
- In this setting, it is still not clear when a Levi-Civita connection exists.
- Although, we've developed quite some theory over the years, I would like to present a case study which shows that even in a very simple case, there are several aspects that come into play.

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- There is usually no “space” anymore, only an algebra.
- One tries to formulate geometric objects in an algebraic way, so that it allows for a generalization to noncommutative algebras.

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- Given two sections $X, Y : M \to E$ one can add them, and multiply by functions

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- That is, the space of sections of a vector bundle is a module over the algebra of functions $C^\infty(M)$.

Vector bundles and projective modules

Because of the following theorem by Serre and Swan (here in the form of Swan) one has a good algebraic notion of a vector bundle.

Theorem (R. G. Swan)

Let X be a compact Hausdorff space, and let $C(X)$ be the ring of continuous functions from X to $\mathbb{R}$. A $C(X)$-module P is isomorphic to a module of sections of a vector bundle if and only if it is a finitely generated projective module.

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Hence, one simply defines a vector bundle over an arbitrary (noncommutative) algebra $\mathcal{A}$ as a finitely generated projective $\mathcal{A}$-module.

Vector bundle $\leftrightarrow$ Finitely generated projective module

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- Yet another way is to choose a distinguished set of derivations on the algebra, defining the calculus.
- In noncommutative geometry, we have to live with the fact that there are many possible choices of differential calculus over an algebra.

Derivation based differential calculus

- In this approach (pioneered by Michel Dubois-Violette), one starts by choosing an algebra $\mathcal{A}$ together with a Lie algebra $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$.
- Is $\mathfrak{g} = \mathrm{Der}(\mathcal{A})$ a canonical choice? Not always, a noncommutative algebra has plenty of inner derivations $\partial(a) = [a, D]$ for some $D \in \mathcal{A}$.
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- For several reasons, one is usually more interested in outer derivations.
- Now, let us start with the pair $(\mathcal{A}, \mathfrak{g})$ and build a differential graded algebra.
- The algebra $\mathcal{A}$ correspond to the "functions" and the Lie algebra $\mathfrak{g}$ correspond to the "vector fields". The differential graded algebra will correspond to the "differential forms".

Given $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$, one defines $\bar{\Omega}^k_{\mathfrak{g}}$ to be the set of $Z(\mathcal{A})$-multilinear alternating maps ($Z(\mathcal{A})$ =center of $\mathcal{A}$)

$$\omega : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k} \to \mathcal{A},$$

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and one gives $\bar{\Omega}^k_{\mathfrak{g}}$ the structure of a $\mathcal{A}$-bimodule by setting

$$\begin{aligned}(a\omega)(\partial_1, \ldots, \partial_k) &= a\omega(\partial_1, \ldots, \partial_k)\\ (\omega a)(\partial_1, \ldots, \partial_k) &= \omega(\partial_1, \ldots, \partial_k)a\end{aligned}$$

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for $a \in \mathcal{A}$, $\omega \in \bar{\Omega}^k_{\mathfrak{g}}$ and $\partial_1, \ldots, \partial_k \in \mathfrak{g}$.
Furthermore, for $\omega \in \bar{\Omega}^k_{\mathfrak{g}}$ and $\tau \in \bar{\Omega}^l_{\mathfrak{g}}$ one defines $\omega\tau \in \bar{\Omega}^{k+l}_{\mathfrak{g}}$ as

$$(\omega\tau)(\partial_1, \ldots, \partial_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\omega(\partial_{\sigma(1)}, \ldots, \partial_{\sigma(k)})\tau(\partial_{\sigma(k+1)}, \ldots, \partial_{\sigma(k+l)}),$$

where S_N denotes the symmetric group on N letters.

For $a \in \mathcal{A}$ one defines $d_0 : \mathcal{A} = \bar{\Omega}^0_{\mathfrak{g}} \to \bar{\Omega}^1_{\mathfrak{g}}$ as

$$(d_0 a)(\partial) = \partial a$$

and for $\omega \in \bar{\Omega}^k_{\mathfrak{g}}$ (for $k \geq 1$) one defines $d_k : \bar{\Omega}^k_{\mathfrak{g}} \to \bar{\Omega}^{k+1}_{\mathfrak{g}}$ by

$$\begin{aligned} d_k\omega(\partial_0, \ldots, \partial_k) &= \sum_{i=0}^{k}(-1)^i \partial_i\big(\omega(\partial_0, \ldots, \hat{\partial}_i, \ldots, \partial_k)\big) \\ &\quad + \sum_{0 \leq i < j \leq k} (-1)^{i+j}\omega\big([\partial_i, \partial_j], \partial_0, \ldots, \hat{\partial}_i, \ldots, \hat{\partial}_j, \ldots, \partial_k\big), \end{aligned}$$

satisfying $d_{k+1}d_k = 0$, where $\hat{\partial}_i$ denotes the omission of ∂_i in the argument. When there is no risk for confusion, we shall omit the index k and simply write $d : \bar{\Omega}^k_{\mathfrak{g}} \to \bar{\Omega}^{k+1}_{\mathfrak{g}}$.

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Since $d^2 = 0$ there is a natural cohomology theory

$$H^k(\Omega_{\mathfrak{g}}) = \ker(d_k) / \operatorname{im}(d_{k-1})$$

Connections and curvature

Let $\mathfrak{g}$ be a Lie subalgebra of $\mathrm{Der}(\mathcal{A})$.

Definition

Let M be a left $\mathcal{A}$-module. A *left connection on M* is a map $\nabla : \mathfrak{g} \times M \to M$ such that

$$\begin{aligned}
\nabla_\partial(m + m') &= \nabla_\partial m + \nabla_\partial m' \\
\nabla_{\partial+\partial'} m &= \nabla_\partial m + \nabla_{\partial'} m \\
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for $m, m' \in M$, $\partial, \partial' \in \mathfrak{g}$, $a \in \mathcal{A}$ and $z \in Z(\mathcal{A})$.

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for $m, m' \in M$, $\partial, \partial' \in \mathfrak{g}$, $a \in \mathcal{A}$ and $z \in Z(\mathcal{A})$.

The curvature of ∇ is the map $R : \mathfrak{g} \times \mathfrak{g} \times M \to M$ defined as

$$R(\partial, \partial')m = \nabla_{\partial}\nabla_{\partial'} m - \nabla_{\partial'}\nabla_{\partial} m - \nabla_{[\partial,\partial']} m.$$

Hermitian forms on modules

The analogue of a metric on a vector bundle is a hermitian form. Compare with

$$h(X, Y) = g(X, \bar{Y})$$

on the complexified tangent bundle.

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Let M be a left $\mathcal{A}$-module. A map $h : M \times M \to \mathcal{A}$ is called a hermitian form on M if

$$\begin{aligned} h(m_1 + m_2, m_3) &= h(m_1, m_3) + h(m_2, m_3) \\ h(am_1, m_2) &= ah(m_1, m_2) \\ h(m_1, m_2)^* &= h(m_2, m_1). \end{aligned}$$

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Metric connections

In Riemannian geometry, a connection is compatible with the metric if

$$X\big(g(m_1, m_2)\big) = g(\nabla_X m_1, m_2) + g(m_1, \nabla_X m_2)$$

for $m_1, m_2 \in M$ and X a vector field.

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for $m_1, m_2 \in M$ and X a vector field. Similarly, one defines a connection on a left $\mathcal{A}$-module to be compatible with a hermitian form h if

$$\partial h(m_1, m_2) = h\big(\nabla_\partial m_1, m_2\big) + h\big(m_1, \nabla_{\partial^*} m_2\big)$$

where $\partial^*(a) = \big(\partial(a^*)\big)^*$.

Levi-Civita connections on $\Omega^1_{\mathfrak{g}}$

Definition

The *torsion* of a left connection ∇ on $\Omega^1_{\mathfrak{g}}$ is given by the map $T : \Omega^1_{\mathfrak{g}} \times \mathfrak{g} \times \mathfrak{g} \to \mathcal{A}$, defined by

$$T_\omega(\partial, \partial') = (\nabla_\partial \omega)(\partial') - (\nabla_{\partial'} \omega)(\partial) - d\omega(\partial, \partial'). \tag{1}$$

The connection is called *torsion free* if $T_\omega(\partial, \partial') = 0$ for all $\partial, \partial' \in \mathfrak{g}$ and $\omega \in \Omega^1_{\mathfrak{g}}$.

Definition

Let h be a left hermitian form on $\bar{\Omega}_{\mathfrak{g}}$. A *left Levi-Civita connection* ∇ *on* $\bar{\Omega}_{\mathfrak{g}}$ *with respect to* h is a torsion free left connection on $\bar{\Omega}_{\mathfrak{g}}$ compatible with h.

The Kronecker algebra

As an example, we would like to study the path algebra originating from the Kroncker quiver.

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_n"'] \arrow[r, phantom, "\vdots"] & 2 \end{tikzcd}$$

Let $\mathcal{K}_N$ denote the unital $\mathbb{C}$-algebra generated by $e, \alpha_1, \ldots, \alpha_N$ satisfying

$$e^2 = e \qquad e\alpha_k = \alpha_k \qquad \alpha_k e = 0 \qquad \alpha_j \alpha_k = 0 \tag{2}$$

for $j, k \in \{1, \ldots, N\}$. The algebra $\mathcal{K}_N$ is finite dimensional, and every element $a \in \mathcal{K}_N$ can be uniquely written as

$$a = \lambda \mathbb{1} + \mu e + a^i \alpha_i$$

for $\lambda, \mu, a^i \in \mathbb{C}$.

Derivations

Proposition

A basis of $\mathrm{Der}(\mathcal{K}_N)$ *is given by* $\{\partial_k\}_{k=1}^N$ *and* $\{\partial_k^l\}_{k,l=1}^N$ *with*

$$\partial_k(e) = i\alpha_k \qquad \partial_k(\alpha_l) = 0$$
$$\partial_k^l(e) = 0 \qquad \partial_k^l(\alpha_j) = \delta_j^l \alpha_k,$$

satisfying

$$[\partial_i^j, \partial_k^l] = \delta_k^j \partial_i^l - \delta_i^l \partial_k^j$$
$$[\partial_i^j, \partial_k] = \delta_k^j \partial_i$$
$$[\partial_i, \partial_j] = 0.$$

Moreover, ∂_i, ∂_i^j *are hermitian derivations for* $i, j = 1, \ldots, N$.

The bimodule of 1-forms $\Omega^1_{\mathfrak{g}}$ is generated by $de, d\alpha_1, \ldots, d\alpha_N$. However, depending on the choice of $\mathfrak{g}$, they might not constitute a basis of $\Omega^1_{\mathfrak{g}}$. To simplify the notation, we set $d\alpha_0 = de$.

Proposition

For any $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{K}_N)$ the bimodule structure of $\Omega^1_{\mathfrak{g}}$ is given by

$$\begin{aligned} e d\alpha_I &= d\alpha_I & \qquad \alpha_i d\alpha_I &= 0 \\ (d\alpha_I)e &= 0 & \qquad (d\alpha_I)\alpha_i &= 0, \end{aligned}$$

for $i = 1, \ldots, N$ and $I = 0, 1, \ldots, N$.

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If $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{K}_N)$ then $H^1(\Omega_{\mathfrak{g}}) = 0$.

Proposition

Let ∇ *be a* $\mathbb{C}$*-bilinear map*

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Note that, due to the specific bimodule structure on $\Omega^1_{\mathfrak{g}}$, there exists a trivial connection. That is, there exists a connection ∇ such that

$$\nabla_{\partial}\omega = 0$$

for all $\omega \in \Omega^1_{\mathfrak{g}}$ and $\partial \in \mathfrak{g}$.

All derivations

Let $\mathfrak{g} = \mathrm{Der}(\mathcal{A})$. In this case, $d\alpha_0 = de, d\alpha_1, \ldots, d\alpha_N$ is a vector space basis of $\mathfrak{g}$.

Proposition

If ∇ is a torsion free connection on Ω^1_{Der} then $\nabla_\partial \omega = 0$ for all $\partial \in \mathrm{Der}(\mathcal{K}_N)$ and $\omega \in \Omega^1_{\mathrm{Der}}$.

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For a torsion free connection to be compatible with a hermitian form, one needs $\partial h(d\alpha_I, d\alpha_J) = 0$ for $I, J = 0, \ldots, N$ and $\partial \in \mathrm{Der}(\mathcal{A})$ implying that

$$h(d\alpha_I, d\alpha_J) = \lambda_{IJ}\mathbb{1}$$

for $\lambda_{IJ} \in \mathbb{C}$.

Outer derivations

Let

$$\mathfrak{g} = \mathbb{C}\left\langle \tilde{\partial}_i = \partial_i + \partial_i^i : \ i = 1, \dots, N \right\rangle .$$

Each $\tilde{\partial}_i$ is an outer derivation and it follows that $d\alpha_1, \dots, d\alpha_N$ is a vector space basis for $\Omega^1_{\mathfrak{g}}$ and $d\alpha_1 + \cdots + d\alpha_N = -ide$.

Proposition

A connection $\nabla : \mathfrak{g} \times \Omega^1_{\mathfrak{g}} \to \Omega^1_{\mathfrak{g}}$ is torsion free if and only if there exists $\gamma_{ij} \in \mathbb{C}$ such that

$$\nabla_{\tilde{\partial}_i} d\alpha_j = \gamma_{ij} d\alpha_i \tag{3}$$

for $i, j = 1, \dots, N$.

Now, let us construct a torsion free connection on $\Omega^1_{\mathfrak{g}}$ that is compatible with the hermitian form given by

$$h_{ij} = h(d\alpha_i, d\alpha_j) = \delta_{ij}\lambda_i\alpha_i$$

for $\lambda_i \in \mathbb{R}$.

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for $\lambda_i \in \mathbb{R}$. Setting

$$\nabla_{\tilde{\partial}_i} d\alpha_j = \tfrac{1}{2}\delta_{ij} d\alpha_i \tag{4}$$

it follows from the previous proposition that ∇ is torsion free. Moreover, one checks that ∇ is compatible with h:

$$\begin{aligned}\tilde{\partial}_i h_{jk} &- h(\nabla_{\tilde{\partial}_i} d\alpha_j, d\alpha_k) - h(d\alpha_j, \nabla_{\tilde{\partial}_i} d\alpha_k) \\ &= \delta_{jk}\lambda_j\tilde{\partial}_i d\alpha_j - \tfrac{1}{2}\delta_{ij}h_{ik} - \tfrac{1}{2}\delta_{ik}h_{ji} \\ &= \lambda_j\delta_{jk}\delta_{ij}\alpha_i - \tfrac{1}{2}\delta_{ij}\delta_{ik}\lambda_i\alpha_i - \tfrac{1}{2}\delta_{ik}\delta_{ji}\lambda_j\alpha_j = 0.\end{aligned}$$

Inner derivations

The Lie algebra of inner derivations is given by

$$\mathfrak{g} = \mathbb{C}\left\langle \partial_1, \ldots, \partial_N, \hat{\partial} = \partial_1^1 + \cdots + \partial_N^N \right\rangle$$

and it follows that $d\alpha_0 = de, d\alpha_1, \ldots, d\alpha_N$ is a basis for $\mathfrak{g}$.

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Proposition

∇ is a torsion free connection on $\Omega^1_{\mathfrak{g}}$ if and only if there exists $\gamma_I^J \in \mathbb{C}$, for $I, J = 0, \ldots, N$ such that

$$\nabla_{\partial_k} d\alpha_I = i\gamma_I^0 d\alpha_k \tag{5}$$

$$\nabla_{\hat{\partial}} d\alpha_I = \gamma_I^J d\alpha_J \tag{6}$$

for $k = 1, \ldots, N$ and $I = 0, \ldots, N$.

Let us now show that there are indeed torsion free connections on $\Omega^1_{\mathfrak{g}}$ compatible with hermitian forms of the type

$$h(d\alpha_I, d\alpha_J) = \gamma_I^0 \gamma_J^0 h_0$$

for arbitrary $h_0 \in \mathbb{C}\langle \alpha_1, \ldots, \alpha_N \rangle$ and $\gamma_I^0 \in \mathbb{R}$. A torsion free connection compatible with h is then given by

$$\begin{aligned} \nabla_{\partial_k} d\alpha_I &= i\gamma_I^0 d\alpha_k \\ \nabla_{\hat{\partial}} d\alpha_I &= \gamma_I^0 de + \frac{\gamma_I^0}{\gamma_{i_0}^0}\left(\tfrac{1}{2} - \gamma_0^0\right) d\alpha_{i_0} \end{aligned}$$

for arbitrary $1 \leq i_0 \leq N$ such that $\gamma_{i_0}^0 \neq 0$.

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for arbitrary $1 \le i_0 \le N$ such that $\gamma_{i_0}^0 \ne 0$.

Thank you for your attention!