A new approach to noncommutative Riemannian spin geometry

I The classical picture

- Let M be a compact oriented Riemannian manifold of dim n
- let TM be the tangent bundle
- there is a canonical principal $GL(n)$-bundle associated TM, the so-called Frame bundle over M
- orientability + Riem. metric $\Rightarrow$ $Fr(M)$ can be reduced to a $SO(n)$-bundle
- $Fr(M) \times_{SO(n)} \mathbb{R}^n \cong TM$
- $spin(n) :=$ universal covering of $SO(n)$

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1$$

- **Def** A spin structure on M is a principal $spin(n)$ - bundle P over M together with a 2-fold covering $\varphi : P \to Fr(M)$ s.t.

$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & Fr(M) \arrow[dl] \curvearrowleft SO(n) \\ & M & \end{tikzcd}$$

- Let $\pi : spin(n) \to Gl(\Delta_n)$ be the so-called spin-repr. on the space Δ_n of n-spinors

- $S := P \times_{spin(n)} \Delta_n$, the so-called spinor bundle over M

- S is hermitian, equipped with an inner product $\langle \cdot , \cdot \rangle$ $\mathcal{H}_s := \overline{\Gamma^{\infty}(S)}^{\langle \cdot , \cdot \rangle}$

smooth-sections

- $C^\infty(M)$ admits a faithful $*$-repr. on $\mathcal{H}_S$ by pointwise multiplication

- There is an unbounded s.a. operator D on $\mathcal{H}_S$ with compact resolvent and bounded commutators $[D, f]$, $f \in C^\infty(M)$

- $(C^\infty(M), \mathcal{H}_S, D)$ is called a spectral triple

D: Dirac operator

II) The NC picture

- Let B be a unital C^*-algebra

- By a spectral triple on B we mean a $\mathbb{D}_B := (B_0, \mathcal{H}, D)$ s.t.

1) $\mathcal{B}_0$ is a dense unital $*$-sub. alg of $\mathcal{B}$

2) $\mathcal{H}$ is a Hilbert space carrying a faithful $*$-repr. of $\mathcal{B}$

3) D is an unbounded s.a. operator with compact resolvent and bounded commutators $[D, b]$, $b \in \mathcal{B}_0$

THM (Connes 08)

$(\mathcal{B}_0, \mathcal{H}, D)$ commutative spectral triple + extra conditions $\Rightarrow$

$$(\mathcal{B}_0, \mathcal{H}, D) \cong (C^\infty(M), \mathcal{H}_S, D)$$

- $$\Omega^1(P_B) = \{ x[D, y] : x, y \in \mathcal{B}_0 \} \subseteq B(\mathcal{H})$$

Connes' differential 1-form

$\cong$ tangent space

III Research in progress

- Let $D_B = (B_0, \mathcal{H}, D)$ be a spectral triple
- Let n be the metric dimension

Hilbert B-bimodule

Q1) (Framing)

NC frame

Is there systematic construction of a NCPB $(A, SO(n), \alpha)$ with $A^{SO(n)} = B$ such that

$$\Omega^1(D_A)^{SO(n)} \cong \Omega^1(D_B)$$

Q2) (Spin)

When is there a 2-fold covering $(C, Spin(n), \gamma)$ of $(A, SO(n), \alpha)$ s.t.

a suitable induction recovers $\mathcal{H}$

in the spectral triple.

THANK YOU!