Hilbert's basis theorem for hom-associative Ore extensions

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Describes mostly joint work with Per Bäck. Johan Öinert, Patrik Lundström and Sergei Silvestrov, among others, have contributed to papers cited.

Rings are unital

All rings in this talk are unital, i.e. there is an element 1 such that $1a = a1 = a$. Rings will be associative at the start of the lecture, at some point we generalize to non-associative rings.

Hilbert's basis theorem

Theorem
If R is an associative, Noetherian ring then $R[x]$ is Noetherian.

Here $R[x]$ is ring generated by R and x subject to the relations $xr = rx$ for every $r \in R$.

Hilbert's basis theorem for Ore extensions

Theorem

If R is an associative, Noetherian ring, σ is an automorphism of R and δ is a σ-derivation then $R[x;\sigma,\delta]$ is a also Noetherian.

By $R[x;\sigma,\delta]$ we mean the associative ring generated by R and x, subject to the relations $xr = \sigma(r)x + \delta(r)$ for every $r \in R$. That δ is a σ-derivation means that

- $\delta(a+b) = \delta(a) + \delta(b)$;
- $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$.

In general σ could be an endomorphism, though Hilbert's basis theorem does not have to hold then.

Ore extensions

The rings $R[x; \sigma, \delta]$ are called Ore extensions, after Norwegian mathematician Øystein Ore who introduced them under the name *non-commutative polynomial rings*.

$R[x; \sigma, \delta]$ is a free left R-module with basis $1, x, x^2, \ldots$.

Degree

Can measure the *degree* of elements in an Ore extension in the same way as in the polynomial ring. Eg $\deg(x^2 - 3x) = 2$.

$$\deg(ab) = \deg(a) + \deg(b)$$

if σ injective and R does not contain zero-divisors.

Examples

Example

If $\sigma = \mathrm{id}_R$ and $\delta = 0$ then $R[x; \sigma, \delta]$ is isomorphic to $R[x]$, the polynomial ring in one central indeterminate.

Example

If $\sigma = \mathrm{id}_R$ then $R[x; \mathrm{id}_R, \delta]$ is a ring of differential polynomials.

Example

If $\delta = 0$ then $R[x; \sigma, 0]$ is a skew polynomial ring.

Examples II

Example

Take $R = k[y]$, $\sigma = \mathrm{id}$ and $\delta(y) = 1$. Then $R[x; \sigma, \delta]$ is the ordinary Weyl algebra.

Example

Take $R = k[y]$, $\sigma(p(y)) = p(qy)$, where $q \in k \setminus \{0, 1\}$ and $\delta(y) = 1$. Then $R[x; \sigma, \delta]$ is called the q-Weyl algebra.

As background to hom-associative Ore extensions we will describe non-associative Ore extensions.

Non-associative rings

By a non-associative ring we mean a not necessarily associative ring. Must be unital and distributive.

Construction

Let σ and δ be additive maps such that $\sigma(1) = 1$ and $\delta(1) = 0$. We equip $R[X]$ with a new multiplication.

The ring structure on $R[X; \sigma, \delta]$ is defined on monomials by

$$aX^m \cdot bX^n = \sum_{i \in \mathbb{N}} a\pi_i^m(b)X^{i+n}, \tag{1}$$

for $a, b \in R$ and $m, n \in \mathbb{N}$, where π_i^m denotes the sum of all the $\binom{m}{i}$ possible compositions of i copies of σ and $m - i$ copies of δ in arbitrary order.

Further generalization

Lundström, Öinert and Richter have further generalized non-associative Ore extension to monoid Ore extensions [6].

Hom-associative rings

A hom-associative ring (R, α) is a non-associative ring R together with an additive function $\alpha : R \to R$ such that

$$\alpha(a)(bc) = (ab)\alpha(c)$$

for all $a, b, c \in R$.
Introduced by Makhlouf and Silvestrov. Connection with Hom-Lie algebras which were defined earlier by Hartwig, Larsson and Silvestrov.
Include the non-associative rings as a special case with $\alpha \equiv 0$.

Hom-associative Ore extensions

Hom-associative Ore extensions were introduced by Bäck, Richter and Silvestrov.
Can ask when non-associative Ore extension can be made into Hom-associative. Some messy conditions in general. Nice special case.

If (R, α) is a hom-associative ring, σ is an endomorphism, δ is a σ-derivation, and α commutes with σ and δ then $(R[x; \sigma, \delta], \alpha)$ is hom-associative, where α has been extended as
$\alpha(\sum a_i x^i) = \sum \alpha(a_i) x^i$.

Non-unitality

Unitality is perhaps not so natural when studying hom-associative rings so our definition of hom-associative Ore extensions is actually for non-unital rings. However for purposes of this talk it is the unital case that is interesting.

Hom-ideals

In hom-associative ideals we are interested in hom-ideals, which are ideals that are invariant under the twisting map α.
Using hom-ideals we can define hom-Noetherian rings in the obvious way.

Hilbert basis theorem

Theorem (Bäck and R.)

Let $\alpha\colon R \to R$ be the twisting map of a unital, hom-associative ring R, and extend the map homogeneously to $R[X;\sigma,\delta]$. Assume further that α commutes with δ and σ, and that σ is an automorphism and δ a σ-derivation on R. If R is right (left) hom-noetherian, then so is $R[X;\sigma,\delta]$.

Non-associative Hilbert basis theorem

Corollary

Let R be a unital, non-associative ring, σ an automorphism and δ a σ-derivation on R. If R is right (left) noetherian, then so is $R[X;\sigma,\delta]$.

Quaternion example

Given a unital and associative algebra A with product $\cdot$ over a field of characteristic different from two, one may define a unital and non-associative algebra A^+ by using the *Jordan product* $\{\cdot,\cdot\}\colon A^+ \to A^+$ given by $\{a,b\} := \frac{1}{2}(a\cdot b + b\cdot a)$ for any $a, b \in A$. A^+ is then a *Jordan algebra*, i.e. a commutative algebra where any two elements a and b satisfy the *Jordan identity*, $\{\{a,b\}\{a,a\}\} = \{a,\{b,\{a,a\}\}\}$.

Example

Let σ be the automorphism on $\mathbb{H}$ defined by $\sigma(i) = -i$, $\sigma(j) = k$, and $\sigma(k) = j$. Any automorphism on $\mathbb{H}$ is also an automorphism on $\mathbb{H}^+$, and hence $\mathbb{H}^+[X;\sigma,0_{\mathbb{H}}]$ is a unital, non-associative Ore extension where e.g. $X\cdot i = -iX$, $X\cdot j = kX$, and $X\cdot k = jX$. $\mathbb{H}^+[X;\sigma,0_{\mathbb{H}}]$ is then noetherian.

Hom-example

Example

This example is adapted from an example in [3].
Let R and S be associative, commutative, and unital rings, and $f\colon R \to S$ a homomorphism. Further assume that R is noetherian. Let A be a non-associative, non-unital, noetherian S-algebra, and define a multiplication $\cdot$ on $U := R \times A$ by $(r_1, a_1) \cdot (r_2, a_2) := (r_1 r_2, f(r_1)a_2 + f(r_2)a_1 + a_1 a_2)$ for any $r_1, r_2 \in R$ and $a_1, a_2 \in A$.

U is then unital with identity element $(1, 0)$, and by defining a twisting map α on U by $\alpha(r, a) := (pr, 0)$ for any $r \in R$, $a \in A$, and $p \in \ker f$, U is hom-associative. Moreover, U is noetherian, and if A is not associative, then U is not associative. Now, let σ_A be an automorphism on A. Then σ defined by $\sigma(r, a) := (r, \sigma_A(a))$ is an automorphism on U. Moreover, if δ_A is a σ_A-derivation on A, then δ defined by $\delta(r, a) := (0, \delta_A(a))$ is a σ-derivation on U, and both δ and σ commute with α. Hence, by previous result, $U[X; \sigma, \delta]$ is noetherian.

Here, one could e.g. take $R = \mathbb{R}[Y]$, $S = \mathbb{R}$, $f\colon \mathbb{R}[Y] \to \mathbb{R}$ the evaluation homomorphism at zero, $p \in \mathbb{R}[Y]$ any polynomial without a constant term, and A, σ_A, and δ_A any $\mathbb{R}$-algebra, σ, and δ, respectively, from the previous examples.

Another example

Lars Hellström contributed this idea.

Let R be a unital, non-associative, noetherian ring, and denote by I the ideal of R generated by all expressions of the form $r(st) - (rs)t$ where $r, s, t \in R$. Define $S := R/I$ and let $\pi\colon R \to S$ be the natural homomorphism. Set $U := R \times S$ and define a multiplication $\cdot$ on U by $(r_1, s_1) \cdot (r_2, s_2) := (r_1 r_2, \pi(r_1)s_2 + s_1\pi(r_2) + s_1 s_2)$ for all $r_1, r_2 \in R$ and $s_1, s_2 \in R$. U is unital with identity element $(1, 0)$, and the map α defined by $\alpha(r, s) := (0, \pi(r) + s)$ for all $r \in R$ and $s \in S$ is a well-defined twisting map that makes U hom-associative.

R, S and U are noetherian. Moreover, if R is not associative, then U is not associative. Now, let σ_R be an endomorphism on R. Then $\sigma_R(I) \subseteq I$, which guarantees the naturally extended endomorphism σ_S on S to be well-defined. By defining $\sigma(r, s) := (\sigma_R(r), \sigma_S(s))$, we get an endomorphism σ on U. Similarly, any σ_R-derivation δ_R satisfies $\delta_R(I) \subseteq I$, and hence the naturally extended σ_S-derivation δ_S is well-defined, and in turn gives rise to a σ-derivation δ on U defined by $\delta(r, s) := (\delta_R(r), \delta_S(s))$. Now, assume that σ_R is an automorphism. Then it is clear that σ_S is surjective. Moreover, we can conclude that $I \subseteq \sigma_R(I)$, which in turn implies that σ_S is injective. Hence σ is an automorphism. Moreover, α commutes with both δ and σ, and therefore $U[X; \sigma, \delta]$ is noetherian.

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