Finite Noncommutative Spaces and Projective Real Calculi

Axel Tiger Norkvist

Department of Mathematics, Linköping University

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Overview

Introduction

Noncommutative Geometry seeks to axiomatize differential geometry so it can be generalized to the study of noncommuative algebras. One popular approach is the use of spectral triples, but there are other approaches as well. A derivation-based approach tries to capture a generalized notion of the connection between vector fields and derivations.

Inspired by a 2013 article by Rosenberg [Ros13], Arnlind and Wilson introduced a framework called real calculi in an effort to introduce Riemannian geometry to noncommutative algebras [AW17b],[AW17a]. This was further developed by Arnlind and Tiger Norkvist by introducing morphisms between real calculi, enabling the study of noncommutative embeddings [ATN20].

Here we discuss how these morphisms are used to understand how free and projective real calculi are connected, and we briefly discuss how finite noncommutative spaces can be modeled using this framework.

Real Calculi, Definition

A real calculus is a structure $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$, where

- $\mathcal{A}$ is a unital *-algebra,
- $\mathfrak{g}$ is a real Lie algebra and $D : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$ is a faithful representation of $\mathfrak{g}$ as a set of hermitian derivations,
- M is a (right) $\mathcal{A}$-module, and
- $\varphi : \mathfrak{g} \to M$ is a $\mathbb{R}$-linear map such that $\varphi(\mathfrak{g})$ generates M.

Let Σ be a smooth manifold. With

- $\mathcal{A} = \mathcal{C}^\infty(\Sigma)$,
- $\mathfrak{g} = \mathrm{Der}(\mathcal{C}^\infty(\Sigma))$ and $D = \mathrm{id}_{\mathfrak{g}}$,
- $M = \mathfrak{X}(\Sigma)$ (the module of smooth vector fields over Σ), and
- $\varphi =$ the natural isomorphism between smooth vector fields and derivations,

we have that $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$ is a real calculus.

Real Calculus Homomorphisms

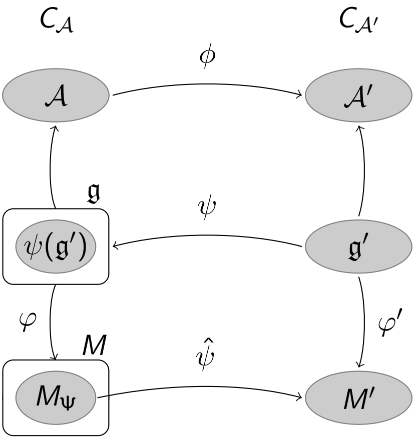
Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$ and $C_{\mathcal{A}'} = (\mathcal{A}', \mathfrak{g}'_{D'}, M', \varphi')$ be two real calculi. $(\phi, \psi, \hat{\psi})$ is said to be a real calculus homomorphism from $C_{\mathcal{A}}$ to $C_{\mathcal{A}'}$ if the following conditions are satisfied:

1. $\phi : \mathcal{A} \to \mathcal{A}'$ is a *-algebra homomorphism,
2. $\psi : \mathfrak{g}' \to \mathfrak{g}$ is a Lie homomorphism such that $\delta(\phi(a)) = \phi(\psi(\delta)(a))$ for all $\delta \in \mathfrak{g}'$ and $a \in \mathcal{A}$.
3. M_Ψ is the submodule of M generated by $\varphi(\psi(\mathfrak{g}'))$, and $\hat{\psi} : M_\Psi \to M'$ is a map that satisfies
   - $\hat{\psi}(m_1 + m_2) = \hat{\psi}(m_1) + \hat{\psi}(m_2)$ for all $m_1, m_2 \in M_\Psi$,
   - $\hat{\psi}(m \cdot a) = \hat{\psi}(m) \cdot \phi(a)$ for all $m \in M_\Psi$ and $a \in \mathcal{A}$, and
   - $\hat{\psi}(\varphi(\psi(\delta))) = \varphi'(\delta)$ for all $\delta \in \mathfrak{g}'$.

Moreover, $(\phi, \psi, \hat{\psi})$ is said to be a real calculus isomorphism if ϕ and ψ are isomorphisms, and $\hat{\psi}$ is a bijection.

Illustration of a Real Calculus Homomorphism

A schematic picture illustrating the real calculus homomorphism $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$:



Example of Finite Noncommutative Space

Let $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$, and let $\mathfrak{g} \subseteq \mathfrak{sl}_N(\mathbb{C})$ be a Lie algebra of skew-hermitian matrices with basis $\{\delta_1, ..., \delta_n\}$. Since every derivation of $\mathcal{A}$ is inner (i.e., they are of the form $\partial = [\delta, \cdot]$ for a unique $\delta \in \mathfrak{sl}_N(\mathbb{C})$, with ∂ being hermitian iff δ is skew-hermitian) we may take D to be the representation given by $D : \delta \mapsto [\delta, \cdot]$.

If we let $\tilde{M} = \mathcal{A}^n$, and let $\tilde{\varphi}$ be such that $\{\tilde{\varphi}(\delta_1), ..., \tilde{\varphi}(\delta_n)\}$ is a basis of $\mathcal{A}^n$, then $\tilde{C}_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, \mathcal{A}^n, \tilde{\varphi})$ is a so-called free real calculus (i.e., $\tilde{M}$ is free, and any basis of $\mathfrak{g}$ generates a basis of $\tilde{M}$ through $\tilde{\varphi}$).

This can be used to generate real calculi $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, P(\mathcal{A}^n), P \circ \tilde{\varphi})$ where $P : \mathcal{A}^m \to \mathcal{A}^m$ is a projection. A real calculus where M is projective is called projective.

The Connection between Free and Projective Real Calculi

In fact, every real projective calculus $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$ is isomorphic to a projective real calculus obtained as the "projection" of a free real calculus.

Proposition

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$ be a real calculus, where M is projective and $\dim \mathfrak{g} = n$. Then there exists a free real calculus $\tilde{C}_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, \mathcal{A}^n, \tilde{\varphi})$ and a projection $P : \mathcal{A}^n \to \mathcal{A}^n$ such that $(\mathcal{A}, \mathfrak{g}_D, P(\mathcal{A}^n), P \circ \tilde{\varphi}) \simeq C_{\mathcal{A}}$.

Thus, we may develop a theory of projective real calculi by using the additional structure provided by a free real calculus.

Proof of Statement

If $\{\delta_1, ..., \delta_n\}$ is a basis of $\mathfrak{g}$ and $\{\hat{e}_1, ..., \hat{e}_n\}$ is a basis of $\mathcal{A}^n$, then if we let $\tilde{\varphi} : \delta_i \mapsto \hat{e}_i$ we get that $\tilde{C}_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, \mathcal{A}^n, \tilde{\varphi})$ is free. Define $\rho : \mathcal{A}^n \to M$ by $\rho(\hat{e}_k m^k) = \varphi(\delta_k) m^k$. Then, by construction, ρ is an epimorphism.

Since M is projective, ρ splits, meaning that there is a monomorphism $\nu : M \to \mathcal{A}^n$ such that $\rho \circ \nu = \mathrm{id}_M$. With $P := \nu \circ \rho$, we have that $P^2 = (\nu \circ \rho) \circ (\nu \circ \rho) = \nu \circ (\rho \circ \nu) \circ \rho = \nu \circ \rho = P$, and it is easy to verify that $(\mathcal{A}, \mathfrak{g}_D, P(\mathcal{A}^m), P \circ \tilde{\varphi})$ is a projective real calculus.

Proof, continued

Let $\hat{\psi} = P \circ \nu = (\nu \circ \rho) \circ \nu = \nu : M \to P(\mathcal{A}^n)$. Then $\hat{\psi}$ is a module isomorphism with inverse $\hat{\psi}^{-1} = \rho|_{P(\mathcal{A}^n)}$, and it is easy to verify that $(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathfrak{g}}, \hat{\psi}) : C_{\mathcal{A}} \to (\mathcal{A}, \mathfrak{g}_D, P(\mathcal{A}^n), P \circ \tilde{\varphi})$ is a real calculus homomorphism:

$$\hat{\psi}(mA) = (P \circ \nu)(mA) = P(\nu(m)A) = (P \circ \nu)(m)A = \hat{\psi}(m)A$$
$$\hat{\psi}(\varphi(\delta_i)) = \hat{\psi}(\rho(\hat{e}_i)) = (P \circ \nu \circ \rho)(\hat{e}_i) = P^2(\hat{e}_i) = P(\hat{e}_i) = P \circ \tilde{\varphi}(\delta_i).$$

The first line shows that $\hat{\psi}$ is compatible with $\phi = \mathrm{id}_{\mathcal{A}}$, and the second line shows that $\hat{\psi}$ is compatible with $\psi = \mathrm{id}_{\mathfrak{g}}$; the compatibility between $\mathrm{id}_{\mathcal{A}}$ and $\mathrm{id}_{\mathfrak{g}}$ is trivial.

Since $\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathfrak{g}}$ and $\hat{\psi}$ are all isomorphisms, the statement follows. □

Morphisms when $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$

In the special case where $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$ and $\mathfrak{g} \subseteq \mathfrak{sl}_N(\mathbb{C})$, let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, \mathcal{A}^n, \varphi)$ and $C'_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_{D'}, \mathcal{A}^n, \varphi')$ both be free real calculi.

To determine whether $C_{\mathcal{A}}$ and $C'_{\mathcal{A}}$ are isomorphic, it suffices to examine the representations D and D'. To do this, we note (as before) that every derivation can be uniquely identified with the commutator of a trace-free matrix. Using this identification, let $\hat{D} : \mathfrak{g} \to \mathrm{Mat}_N(\mathbb{C})$ and $\hat{D}' : \mathfrak{g} \to \mathrm{Mat}_N(\mathbb{C})$ be the associated matrix representations of $\mathfrak{g}$ with D and D', respectively. (i.e., if $D(\delta) = [d, \cdot]$ then $\hat{D}(\delta) = d$).

The second key observation is the Skolem-Noether theorem, which states that every automorphism of $\mathrm{Mat}_N(\mathbb{C})$ is of the form $\phi = \phi_U$, where U is a nonsingular matrix and $\phi_U(A) = U^{-1}AU$.

Morphisms when $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$, continued

Proposition

$C_{\mathcal{A}} \simeq C'_{\mathcal{A}}$ if and only if there is a nonsingular matrix U and an automorphism ψ of $\mathfrak{g}$ such that

$$\hat{D}'(\delta) = U^{-1}\hat{D}(\psi(\delta))U, \quad \delta \in \mathfrak{g}.$$

Matrix representations that satisfy the above relation are referred to as quasi-equivalent.

Connections

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$ be a real calculus. An affine connection $\nabla : \mathfrak{g} \times M \to M$ is a map that satisfies

- $\nabla_\partial(m+n) = \nabla_\partial m + \nabla_\partial n$ for all $m, n \in M$ and $\partial \in \mathfrak{g}$,
- $\nabla_{\lambda\partial_1+\partial_2} m = \lambda\nabla_{\partial_1} m + \nabla_{\partial_2} m$ for all $m \in M$, $\lambda \in \mathbb{R}$ and $\partial_1, \partial_2 \in \mathfrak{g}$, and
- $\nabla_\partial(m \cdot a) = (\nabla_\partial m) \cdot a + m \cdot \partial(a)$ for all $m \in M$, $\partial \in \mathfrak{g}$ and $a \in \mathcal{A}$.

Proposition

Every projective real calculus has an affine connection.

This statement is proved by using the fact that every projective real calculus is a "projection" of a free calculus $\tilde{C}_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, \mathcal{A}^n, \tilde{\varphi})$, and defining a connection $\tilde{\nabla}$ on $\mathcal{A}^n$. Then $P \circ \tilde{\nabla}$ is a connection on $P(\mathcal{A}^n)$.

Metrics

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$ be a real calculus. A *metric* $h : M \times M \to \mathcal{A}$ is a Hermitian form that is non-degenerate, i.e.

- $h(m_1 + m_2, n) = h(m_1, n) + h(m_2, n)$ for all $m_1, m_2, n \in M$,
- $h(m, n \cdot a) = h(m, n)a$ for all $m, n \in M$, $a \in \mathcal{A}$,
- $h(m, n) = h(n, m)^*$ for all $m, n \in M$, and
- $h(m, n) = 0$ for all $n \in M \Rightarrow m = 0$.

Moreover, if $h(\varphi(\partial_1), \varphi(\partial_2)) = h(\varphi(\partial_1), \varphi(\partial_2))^*$ for all $\partial_1, \partial_2 \in \mathfrak{g}$ (i.e., it is truly symmetric on $\varphi(\mathfrak{g})$) then $(C_{\mathcal{A}}, h)$ is called a real metric calculus.

Metrics and Connections

We now define what it means for a connection to be compatible with the metric, and torsion-free. Let $(C_{\mathcal{A}}, h)$ be a real metric calculus and let $\nabla : \mathfrak{g} \times M \to M$ be a connection.

1. If $h(\varphi(\delta_i), \nabla_{\delta_j}\varphi(\delta_k))$ is hermitian for every $\delta_i, \delta_j, \delta_k \in \mathfrak{g}$, then $(C_{\mathcal{A}}, h, \nabla)$ is said to be a real connection calculus.
2. Moreover, if
$$\delta(h(\varphi(\delta_i), \varphi(\delta_j))) = h(\nabla_\delta \varphi(\delta_i), \varphi(\delta_j)) + h(\varphi(\delta_i), \nabla_\delta \varphi(\delta_j))$$
for every $\delta, \delta_i, \delta_j \in \mathfrak{g}$, then ∇ is said to be metric.
3. Additionally, if
$$\nabla_{\delta_i}\varphi(\delta_j) - \nabla_{\delta_j}\varphi(\delta_i) - \varphi([\delta_i, \delta_j]) = 0$$
for every $\delta_i, \delta_j \in \mathfrak{g}$, then ∇ is said to be torsion-free.

A real connection calculus $(C_{\mathcal{A}}, h, \nabla)$ where ∇ is both metric and torsion-free is said to be pseudo-Riemannian, and ∇ is called the Levi-Civita connection.

Free Real Metric Calculi

Suppose $\tilde{C}_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, \mathcal{A}^n, \tilde{\varphi})$ is a free real calculus, and let $\tilde{h}$ be a metric on $\tilde{M}$ such that $(\tilde{C}_{\mathcal{A}}, \tilde{h})$ is a real metric calculus. If $\tilde{h}$ satisfies the additional condition of being invertible, then we say that $(\tilde{C}_{\mathcal{A}}, \tilde{h})$ is a free real metric calculus.

Although this definition of a free real calculus is rather restrictive, it comes with some advantages. For instance, the Levi-Civita connection always exists uniquely for free real metric calculi.

Metrics and projections

If $\tilde{h}$ is a metric on the free module $\mathcal{A}^n$, and $\tilde{h}$ is orthogonal with respect to the projection $P : \mathcal{A}^n \to \mathcal{A}^n$, then it is clear that the restriction h of $\tilde{h}$ to $P(\mathcal{A}^n)$ is a metric on $P(\mathcal{A}^n)$.

Given a free real metric calculus $(\tilde{C}_{\mathcal{A}}, \tilde{h})$ and a projection P such that $\tilde{h}$ is orthogonal with respect to P, we may let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, P(\mathcal{A}^n), \varphi)$ with $\varphi = P \circ \tilde{\varphi}$ as before. However, if we take $h := \tilde{h}|_{P(\mathcal{A}^n) \times P(\mathcal{A}^n)}$, then it is not guaranteed that $(C_{\mathcal{A}}, h)$ is a real metric calculus due to the symmetry of h needed on $\varphi(\mathfrak{g})$.

The extra condition that $\tilde{h}$ and P must satisfy for $(C_{\mathcal{A}}, h)$ is that $\tilde{h}_{jk} P^k_i = \tilde{h}_{ik} P^k_j$, i.e., hP is symmetric when viewed as a matrix.

Current work

- Current work is mainly connected to understanding whether a projective real metric calculus always has a Levi-Civita connection using the underlying structure of a free real metric calculus. While it is relatively straight-forward to deal with the metric condition, the torsion-freeness is more problematic to deal with.
- We are still looking at the case $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$, and as an example of a projective real calculus we are starting to examine $(\mathbb{C}^N)^n$ as a natural candidate for what constitutes an interesting projective module to highlight the concepts discussed.

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