Hom-algebra structures

AMS MSC 2020: **17B61 Hom-Lie and related algebras,**

17D30 (non-Lie) Hom algebras and topics

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Swedish Network for Algebra and Geometry
SNAG 2020 workshop
September 24 - September 25, 2020
(Date of the talk: September 25, 2020)

Outline

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Motivation

- **Discretizations (quasi-deformations) of vector fields, (quantum) q-deformations of finite-dimensional Lie algebras and infinite-dimensional Lie algebras like Witt and Virasoro algebras, ...** ;
- **q-Deformed vertex operator models of CFT; quantum field theory; quantization**
  1990's – ......
  Lukierski, Kulish, Ellinas, Prischnaider, Isaev,
  Aizawa, Sato, Hu, Liu, Belov, Chaltikian,
  Curtright, Zachos,
  Dobrev, Doebner, Twarock....

More Motivation

- **extensions and q-deformations of differential and homological algebra,**
  differential geometric structures, non-commutative, twisted differential calculi non-commutative quantum field theory
  1990's, 2000's – ......
  V. Abramov, O. Liivapuu, R. Kerner, Dimakis, F. Muller-Hoissen, Lychagin, Huru, Jean-Christophe Wallet, M. Dubois-Violette, Axel de Goursaca, Thierry Massona, Kapranov, Kassel, Kac, Borowec, ...

More motivation

- ***q*-deformed Heisenberg (Weyl) algebras, quantum oscillator algebras, quantum algebras, braided Lie algebras** 1990's ...
  Hellstrom and Silvestrov (book, World Scientific 2000),
  Kulish, Damaskinsky, Fairlie, Curtright, Zachos, Michel Rausch de Traubenberg, ...
  Gurevich, Majid, Lychagin, Huru, ... **(braided Lie algebras)**
- ***q*-analysis, *q*-special functions**
  (1850's –...– 1910's, ... Euler, Gauss, Jackson, ...)

More motivation

- **Color Lie (super)algebras (Γ-graded ϵ-Lie algebras), in particular Lie Super algebras**
  1978 – .... Lukierski, Rittenberg, Wyler, Scheunert, Marcinek, Kwasniewski, Bachturin, Mikhalev, ...
  ... **Sergei Silvestrov (1992, 1994, 1996, 2005: 7 papers, classification, involutoins, representations, ...)**
  2008 – Jean-Christophe Wallet, Axel de Goursaca, Thierry Massona, Michel Rausch de Traubenberg
- **non-associative algebras**
- **non-commutative geometry**
  (algebraic, differential, ...)

σ-derivations

(twisted, deformed, discretized derivations)

$\mathcal{A}$ – (commutative) associative $\mathbb{K}$-algebra with unity

$\sigma : \mathcal{A} \to \mathcal{A}$ algebra endomorphism

σ-derivations

- $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ linear map
- twisted (deformed) Leibniz rule

$$\partial_\sigma(a \cdot b) = \partial_\sigma(a) \cdot b + \sigma(a) \cdot \partial_\sigma(b)$$

Examples of σ-derivations

- the ordinary derivation operator
  $(\partial\, a)(x) = \lim\limits_{y\to x} \frac{a(y)-a(x)}{y-x} = \frac{da}{dx}(x) = a'(x)$
  $(\partial\, ab)(x) = (\partial a)(x)b(x) + a(x)(\partial b)(x)$
  $\sigma = \mathrm{id} : a(x) \mapsto a(x)$
- Shifted difference operators
  $(\partial\, a)(x) = a(x+h) - a(x)$
  $(\partial\, ab)(x) = (\partial a)(x)b(x) + a(x+h)(\partial b)(x)$
  $\sigma(a)(x) = a(x+h)$
- q-difference operator
  $(\partial\, a)(x) = a(qx) - a(x)$
  $(\partial\, ab)(x) = (\partial\, a)(x)b(x) + a(qx)(\partial\, b)(x)$
  $\sigma(a)(x) = a(qx)$

Examples of σ-derivations

- Jackson q-derivative
  $(\partial a)(x) = (D_q a)(x) = \frac{a(qx)-a(x)}{qx-x}$
  $(\partial ab)(x) = (\partial a)(x)b(x) + a(qx)(\partial b)(x)$
  $\sigma(a)(x) = a(qx)$
  $\lim_{q\to 1} D_q(a)(x) = a'(x)$
- **"General" σ-derivations (twisted difference operators)**
  $\Omega \subset \mathbb{K}$ any subset of a field
  $T : \Omega \to \Omega$
  Any transformation without fixed points in Ω
  A any algebra of functions a on Ω such that
  $\sigma(a)(x) = a(T(x)) \in A$
  $\partial_\sigma : a(x) \mapsto \frac{a(T(x))-a(x)}{T(x)-x} = \left(\frac{(\sigma-id)}{(T-id)}a\right)(x)$
  $\Downarrow$
  $\partial_\sigma(a\cdot b) = \partial_\sigma(a)\cdot b + \sigma(a)\cdot\partial_\sigma(b)$

σ-derivations on UFD

Theorem 1 $\mathcal{A}$ is UFD (unique factorization domain)

$$\Downarrow$$

Space of all σ-derivations $\mathfrak{D}_\sigma(\mathcal{A})$ is a free rank one $\mathcal{A}$-module with generator

$$\Delta = \frac{(\mathrm{id} - \sigma)}{g} : \quad a \longmapsto \frac{(\mathrm{id} - \sigma)(a)}{g}$$

where $g = \gcd\left((\mathrm{id} - \sigma)(\mathcal{A})\right)$

Quasi-Hom-Lie algebras of twisted (deformed) vector fields

$\mathcal{A}$ commutative algebra, $\sigma : \mathcal{A} \to \mathcal{A}$ algebra endomorphism, $\Delta : \mathcal{A} \to \mathcal{A}$ σ-derivation,

$Ann(\Delta) = \{a \in A \mid a\Delta = 0\}$, σ-twisted vector fields $\mathcal{A} \cdot \Delta$

Theorem 2 (Hartwig, Larsson, Silvestrov, 2003)

J. of Algebra, 295 (2006), 314-361, Preprint Institute Mittag-Lefler, 2003, Preprint Lund University 2003

Bracket on $\mathcal{A} \cdot \Delta$ (well-defined if $\sigma(Ann(\Delta)) \subseteq Ann(\Delta)$)

$$\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = (\sigma(a) \cdot \Delta)(b \cdot \Delta) - (\sigma(b) \cdot \Delta)(a \cdot \Delta)$$

Closure $\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = \big(\sigma(a)\Delta(b) - \sigma(b)\Delta(a)\big) \cdot \Delta$

Skew-symmetry $\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = -\langle b \cdot \Delta, a \cdot \Delta \rangle_\sigma$

Twisted 6 term Jacobi Identity

If $\Delta \circ \sigma(a) = \delta \cdot \sigma \circ \Delta(a)$, for some $\delta \in A$, then $\forall a, b, c \in \mathcal{A}$:

$$\circlearrowleft_{a,b,c} \Big(\big\langle \sigma(a) \cdot \Delta, \langle b \cdot \Delta, c \cdot \Delta \rangle_\sigma \big\rangle_\sigma + \delta \cdot \big\langle a \cdot \Delta, \langle b \cdot \Delta, c \cdot \Delta \rangle_\sigma \big\rangle_\sigma \Big) = 0$$

$$\mathcal{A} \text{ is UFD} \Rightarrow \quad \delta = \frac{\sigma(g)}{g}, \quad g = GCD(id - \sigma)(\mathcal{A})$$

Quasi-Lie algebra

Quasi-Lie algebras were first introduced in

D. Larsson, S. Silvestrov, Quasi-Lie algebras, In: Jurgen Fuchs, et al. (eds), "Noncommutative Geometry and Representation Theory in Mathematical Physics", American Mathematical Society, Contemporary Mathematics, Vol. 391, 2005.

$(L, \langle\cdot,\cdot\rangle_L, \alpha, \beta, \omega, \theta)$

1. L is a linear space over $\mathbb{F}$,
2. $\langle\cdot,\cdot\rangle_L : L \times L \to L$ is a bilinear product or bracket in L;
3. $\alpha, \beta : L \to L$, are linear maps,
4. $\omega : D_\omega \to \mathcal{L}_{\mathbb{F}}(L)$ and $\theta : D_\theta \to \mathcal{L}_{\mathbb{F}}(L)$ are maps with domains of definition $D_\omega, D_\theta \subseteq L \times L$,

ω-**Symmetry** $\forall (x, y) \in D_\omega$

$$\langle x, y\rangle_L = \omega(x, y)\langle y, x\rangle_L,$$

Quasi-Jacobi identity $\forall (z, x), (x, y), (y, z) \in D_\theta$

$$\circlearrowleft_{x,y,z} \left\{ \theta(z, x)\big(\langle\alpha(x), \langle y, z\rangle_L\rangle_L + \beta\langle x, \langle y, z\rangle_L\rangle_L\big)\right\} = 0$$

Hom-Lie algebras

Hom-Lie algebras is special subclass of Quasi-Lie algebras

$$\beta = 0, \quad \omega = -\operatorname{id}_L, \quad \theta = \operatorname{id}_L$$

1. linear map $\alpha : L \to L$
2. bilinear multiplication (bracket) $\langle \cdot, \cdot \rangle_\alpha$ such that

- **skew-symmetry** $\langle x, y \rangle_\alpha = -\langle y, x \rangle_\alpha$
- **Hom-Lie Jacobi identity** $\forall x, y, z \in L$

$$\circlearrowleft_{x,y,z} \big\langle \alpha(x), \langle y, z \rangle_\alpha \big\rangle_\alpha = 0$$

Quasi-Hom-Lie algebras

$(L, \langle\cdot,\cdot\rangle_L, \alpha, \beta, \omega)$

1. L is a linear space over field $\mathbb{K}$
2. $\langle\cdot,\cdot\rangle_L : L \times L \to L$ is a bilinear map
3. $\alpha, \beta : L \to L$ are linear maps
4. $\omega : D_\omega \to End(L)$ is a map with domain of definition $D_\omega \subseteq L \times L$ taking values in linear operators on L

- (β-**twisting**) α is a β-twisted algebra endomorphism:

$$\langle\alpha(x), \alpha(y)\rangle_L = \beta \circ \alpha\langle x, y\rangle_L \quad \forall x, y \in L$$

- (ω-**symmetry**) $\langle x, y\rangle_L = \omega(x, y)\langle y, x\rangle_L \quad \forall (x, y) \in D_\omega$
- **Quasi-Hom-Lie Jacobi identity** $\forall (z, x), (x, y), (y, z) \in D_\omega$

$$\circlearrowleft_{x,y,z} \left\{ \omega(z, x)\Big(\langle\alpha(x), \langle y, z\rangle_L\rangle_L + \beta\langle x, \langle y, z\rangle_L\rangle_L\Big) \right\} = 0$$

Color (graded) Quasi-Lie algebra

First introduced in:

D. Larsson, S. Silvestrov, Graded quasi-Lie algebras, Czechoslovak J. Phys., 55, 11 (2005), 1473-1478

Γ-graded (color) quasi-Lie algebra $(L, \langle\cdot,\cdot\rangle_L, \alpha, \beta, \omega, \theta)$
$(\Gamma, \hat{+})$ commutative semigroup

1. $L = \bigoplus_{\gamma\in\Gamma} L_\gamma$ is a Γ-graded linear space over $\mathbb{F}$,
2. $\langle\cdot,\cdot\rangle_L : L \times L \to L$ is a bilinear map (bracket);
3. $\alpha, \beta : L \to L$ are linear maps,

$$\alpha(\cup_{\gamma\in\Gamma} L_\gamma) \subseteq \cup_{\gamma\in\Gamma} L_\gamma, \quad \beta(\cup_{\gamma\in\Gamma} L_\gamma) \subseteq \cup_{\gamma\in\Gamma} L_\gamma$$
$$D_\omega, D_\theta \subseteq \cup_{\gamma\in\Gamma} L_\gamma \times \cup_{\gamma\in\Gamma} L_\gamma$$
$$\omega : D_\omega \to \mathcal{L}_{\mathbb{F}}(L), \quad \theta : D_\theta \to \mathcal{L}_{\mathbb{F}}(L)$$

Γ-grading axiom $\forall \gamma_1, \gamma_2 \in \Gamma$: $\langle L_{\gamma_1}, L_{\gamma_2}\rangle_L \subseteq L_{\gamma_1 \hat{+} \gamma_2}$

ω-symmetry $\forall (x,y) \in D_\omega$: $\langle x, y\rangle_L = \omega(x,y)\langle y, x\rangle_L$

quasi-Jacobi identity $\forall (z,x) \in D_\theta, (x,y) \in D_\theta, (y,z) \in D_\theta$

$$\circlearrowleft_{x,y,z} \left\{ \theta(z,x)\big(\langle \alpha(x), \langle y, z\rangle_L\rangle_L + \beta\langle x, \langle y, z\rangle_L\rangle_L\big) \right\} = 0$$

Note: $(\omega(x,y)\omega(y,x) - \mathrm{id})\langle x, y\rangle = 0$, if $(x,y), (y,x) \in D_\omega$ by ω-symmetry

Hom-Lie color algebra

Hom-Lie color algebras is special subclass of color (graded) quasi-Lie algebras first introduced in:

D. Larsson, S. Silvestrov, Graded quasi-Lie algebras, Czechoslovak J. Phys., 55, 11 (2005), 1473-1478

Hom-Lie superalgebras $\Gamma = \mathbb{Z}_2$ and $\varepsilon(a, b) = (-1)^{|a||b|}$
$(L, [\cdot, \cdot], \alpha, \varepsilon)$
L is Γ-graded space
$[\cdot, \cdot] : L \times L \to L$ is an even bilinear mapping
α is an even linear map
ε is bi-character on Γ
ε-**skew-symmetry** $[x, y] = -\varepsilon(x, y)[y, x]$,
Hom ε-Jacobi identity
$\circlearrowleft_{x,y,z} \varepsilon(z, x)[\alpha(x), [y, z]] = 0$,
for all homogenous elements x, y, z in L.

Γ-graded ε-Lie algebras (color Lie)

Γ – commutative group (or semigroup).
K field of $char K \neq 2, 3$
Γ-graded algebra

$$L = \bigoplus_{\gamma \in \Gamma} L_\gamma$$

$\langle \cdot, \cdot \rangle : L \times L \longrightarrow L$ bilinear map

$\forall \quad A \in L_\alpha, B \in L_\beta, C \in L_\gamma, \quad \alpha, \beta, \gamma \in \Gamma$:

$\langle A, B \rangle = -\varepsilon(\alpha, \beta) \langle B, A \rangle$ (ε-skew symmetry)

$\varepsilon(\gamma, \alpha)\langle A, \langle B, C \rangle\rangle + \varepsilon(\beta, \gamma)\langle C, \langle A, B \rangle\rangle + \varepsilon(\alpha, \beta)\langle B, \langle C, A \rangle\rangle = 0$
(ε-Jacoby identity)

Color Lie algebras are examples of quasi Hom-Lie algebras.

L Γ-graded quasi Hom-Lie algebra (color Lie algebra)
$L = \oplus_{\gamma\in\Gamma} L_\gamma$
$\alpha = \beta = \mathrm{id}_L, \quad \omega(x,y)v = -\varepsilon(\gamma_x,\gamma_y)v$
$v \in L$ and $(x,y) \in D_\omega = \cup_{\gamma\in\Gamma} L_\gamma$
$\gamma_x, \gamma_y \in \Gamma$ graded degrees of x and y.
The ω-symmetry and the quasi-hom-Lie-Jacobi identity

$$\downarrow$$

Γ-Graded ε-symmetry and ε-Jacobi identities for color Lie algebras.
Lie superalgebras
$\Gamma = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$,
$\varepsilon(\gamma_x,\gamma_y) = (-1)^{\gamma_x\gamma_y}$
$\gamma_x\gamma_y$ is the product in $\mathbb{Z}_2$.

Quasi-Leibniz algebra

Quasi-Leibniz algebras were first introduced in

D. Larsson, S. Silvestrov, Quasi-Lie algebras, In: Jurgen Fuchs, et al. (eds), "Noncommutative Geometry and Representation Theory in Mathematical Physics", American Mathematical Society, Contemporary Mathematics, Vol. 391, 2005.

Quasi-Leibniz-Loday algebra

For $(z,x),(x,y),(y,z)\in D_\theta$, $(\alpha(z),\langle x,y\rangle),(\alpha(y),\langle z,x\rangle),(y,\langle z,x\rangle),(z,x)\in D_\omega$

$$\begin{aligned}\theta(y,z)\big(\omega(\alpha(z),\langle x,y\rangle)\langle\langle x,y\rangle,\alpha(z)\rangle+\beta\circ\omega(z,\langle x,y\rangle)\langle\langle x,y\rangle,z\rangle\big)=\\ =-\theta(x,y)\big(\omega(\alpha(y),\langle z,x\rangle)\langle\omega(z,x)\langle x,z\rangle,\alpha(y)\rangle+\\ +\beta\circ\omega(y,\langle z,x\rangle)\langle\omega(z,x)\langle x,z\rangle,y\rangle\rangle\big)-\\ -\theta(z,x)\big(\langle\alpha(x),\langle y,z\rangle\rangle+\beta\langle x,\langle y,z\rangle\rangle\big)\end{aligned}$$

For $\alpha=\mathrm{id},\beta=0$ and $\theta=\omega=-\mathrm{id}$, **Leibniz-Loday algebra** $\langle\langle x,y\rangle,z\rangle=\langle\langle x,z\rangle,y\rangle+\langle x,\langle y,z\rangle\rangle$.

Hom-Leibniz algebras (Hom-Leibniz-Loday algebras)

(special subclass of quasi-Leibniz algebras)

Quasi-Leibniz algebras were first introduced in

D. Larsson, S. Silvestrov, Quasi-Lie algebras, In: Jurgen Fuchs, et al. (eds), "Noncommutative Geometry and Representation Theory in Mathematical Physics", American Mathematical Society, Contemporary Mathematics, Vol. 391, 2005.

Hom-Leibniz-Loday algebra is a triple $(V, \langle\cdot,\cdot\rangle, \alpha)$, where
V is a linear space,
$\alpha : V \to V$ is a linear map (linear space endomorphism of V)
$\langle\cdot,\cdot\rangle : V \times V \to V$ is a bilinear map
satisfying:

$$\langle\langle x, y\rangle, \alpha(z)\rangle = \langle\langle x, z\rangle, \alpha(y)\rangle + \langle\alpha(x), \langle y, z\rangle\rangle$$

Hom-Lie algebras are skew-symmetric Hom-Leibniz algebras

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q-Deformed Witt algebra Witt_q

$$\mathfrak{D}_\sigma(\mathcal{A}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C} \cdot d_n$$

$\Delta = tD_q = \frac{\sigma - \mathrm{id}}{q-1} : f(t) \mapsto \frac{f(qt)-f(t)}{q-1}$

$\sigma(t) = qt, \quad \sigma(f)(t) = f(qt), \quad \{n\}_q = \frac{q^n-1}{q-1}$

Skew-symmetric product. $d_n = -t^n\Delta$

$$\langle d_n, d_m\rangle = q^n d_n d_m - q^m d_m d_n = \big(\{n\}_q - \{m\}_q\big) d_{n+m}$$

Graded Hom-Lie algebra $\langle L_n, L_m\rangle \subseteq L_{n+m}$

Hom-Lie algebra Jacobi-identity

$\circlearrowleft_{n,m,l} (q^n+1)\big\langle d_n, \langle d_m, d_l\rangle\big\rangle = 0$

$\alpha(d_n) = (q^n+1)d_n$

$\circlearrowleft_{n,m,l} \big\langle \alpha(d_n), \langle d_m, d_l\rangle\big\rangle = 0$

q-Deformed Virasoro algebra. Hom-Lie central extension

$$(\mathrm{Vir}_q, \hat{\sigma}) = (\mathrm{Witt}_q \oplus \mathbb{C}\cdot \mathsf{c}, \hat{\sigma}) \quad \{d_n : n \in \mathbb{Z}\} \cup \{\mathsf{c}\}$$

$$\hat{\sigma} : \mathrm{Vir}_q \to \mathrm{Vir}_q, \quad \hat{\sigma}(d_n) = q^n d_n, \quad \hat{\sigma}(\mathsf{c}) = \mathsf{c}$$

$$\langle d_n, d_m \rangle = \left(\{n\}_q - \{m\}_q\right) d_{n+m} + \\ +\delta_{n+m,0}\frac{q^{-n}}{6(1+q^n)}\{n-1\}_q\{n\}_q\{n+1\}_q \mathsf{c}$$

$$\langle \mathsf{c}, \mathrm{Vir}_q \rangle = 0$$

Loop quasi Hom-Lie algebras

Quasi-Hom-Lie algebra $\mathfrak{g}$

$$\Downarrow$$

Linear space

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{K}[t, t^{-1}]$$

The algebra of Laurent polynomials with coefficients in the qhl-algebra $\mathfrak{g}$.

$\alpha_{\hat{\mathfrak{g}}} := \alpha_{\mathfrak{g}} \otimes \mathrm{id}$
$\beta_{\hat{\mathfrak{g}}} := \beta_{\mathfrak{g}} \otimes \mathrm{id}$
$\omega_{\hat{\mathfrak{g}}} := \omega_{\mathfrak{g}} \otimes \mathrm{id}$
$\langle x \otimes t^n, y \otimes t^m \rangle_{\hat{\mathfrak{g}}} = \langle x, y \rangle_{\mathfrak{g}} \otimes t^{n+m}$
$\hat{\mathfrak{g}}$ is a quasi Hom-Lie algebra.

Non-linear Quasi-Lie deformations of Witt algebra

$$\mathfrak{D}_\sigma(\mathcal{A}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot d_n,$$

$$D = \alpha t^{-k+1}\frac{\mathrm{id}-\sigma}{t-qt^s}, \quad \sigma(t) = qt^s$$

Skew-symmetric product $\quad d_n = -t^n D$

$$\langle d_n, d_m\rangle_\sigma = q^n d_{ns} d_m - q^m d_{ms} d_n$$

$$\langle d_n, d_m\rangle_\sigma = \text{linear combinations of generators}$$

For $n, m \geq 0$:

$$\langle d_n, d_m\rangle_\sigma = \alpha \operatorname{sign}(n-m) \sum_{l=\min(n,m)}^{\max(n,m)-1} q^{n+m-1-l} d_{s(n+m-1)-(k-1)-l(s-1)}$$

Non-linear Quasi-Lie deformations of Witt algebra

σ-deformed Jacobi-identity

$$\circlearrowleft_{n,m,l} \left(\left\langle q^n d_{ns}, \langle d_m, d_l \rangle_\sigma \right\rangle_\sigma + \underbrace{q^k t^{k(s-1)} \sum_{r=0}^{s-1} (qt^{s-1})^r}_{=\delta} \left\langle d_n \langle d_m, d_l \rangle_\sigma \right\rangle_\sigma \right) = 0.$$

Quasi-Hom-Lie algebra, not Hom-Lie algebra for $s \neq 1$

Other non-linear Quasi-Lie deformations of Witt algebra

$\sigma(t) = qt^s$

$$D = \frac{\mathrm{id} - \sigma}{\eta^{-1} \cdot t^k}$$

generates a cyclic $\mathcal{A}$-submodule $\mathfrak{M}$ of $\mathfrak{D}_\sigma(\mathcal{A})$, proper for $s \neq 1$ ($s \neq 1$: $\sigma(t) = \beta t$ for some $\beta \in \mathbb{K}$)

Theorem The linear space

$$\mathfrak{M} = \bigoplus_{i \in \mathbb{Z}} \mathbb{K} \cdot d_i \quad \text{with} \quad d_i = -t^i D$$

is a quasi-Lie algebra

$$\langle d_n, d_m \rangle_\sigma = q^n d_{ns} d_m - q^m d_{ms} d_n = \eta q^m d_{ms+n-k} - \eta q^n d_{ns+m-k}$$

$s \in \mathbb{Z}$ and $\eta \in \mathbb{C}$

Other non-linear Quasi-Lie deformations of Witt algebra

The σ-deformed Jacobi identity

$$\circlearrowleft_{n,m,l} \left(\left\langle q^n d_{ns}, \langle d_m, d_l \rangle_\sigma \right\rangle_\sigma + \underbrace{q^k t^{(s-1)k}}_{=\delta} \left\langle d_n, \langle d_m, d_l \rangle_\sigma \right\rangle_\sigma \right) = 0$$

$q = 1$, $k = 0$ **and** $s = 1$
get a commutative algebra with countable number of generators instead of the Witt algebra.

Non-linear Quasi-Lie deformations of Witt algebra are almost graded

Almost graded algebras $s = 1$:
$\langle L_n, L_m \rangle_\sigma \subseteq L_{n+m-k}$
(quasi-Lie deformations of) Krichever-Novikov type algebras,
Graded $k = 0, s = 1$: $\langle L_n, L_m \rangle_\sigma \subseteq L_{n+m}$
Hyper almost Graded algebras:

$$\langle L_n, L_m \rangle_\sigma \subseteq \bigoplus_{j \in \mathbb{Z} \cap [ms+n-k, ns+m-k]} L_j$$

$$ms + n - k = m + n + m(s-1) - k$$

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$

Daniel Larsson, Sergei Silvestrov, (2005) Communications in Algebra, 35, 4303-4318, 2007.
D. Larsson, S. Silvestrov, The Lie algebra $sl_2(F)$ and quasi-deformations, Czechoslovak Journal of Physics, **55**, 11 (2005), 1467-1472.
D. Larsson, G. Sigurdsson, S. Silvestrov, On some almost quadratic algebras coming from twisted derivations, J. Nonlin. Math. Phys. Vol. 13, (2006).
(Preprints in mathematical sciences (2006:9), LUTFMA-5073-2006, Centre for Mathematical Sciences, Lund University. 11 pp.)
D. Larsson, S. D. Silvestrov, Quasi-deformations of $sl_2(F)$ with base $\mathbb{R}[t, t^{-1}]$. Czechoslovak J. Phys. 56 (2006), no. 10-11, 1227–1230

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$

$$\mathfrak{sl}_2(\mathbb{K}) : [\mathsf{h}, \mathsf{e}] = 2\mathsf{e},\ [\mathsf{h}, \mathsf{f}] = -2\mathsf{f},\ [\mathsf{e}, \mathsf{f}] = \mathsf{h}$$

Representation

$\mathsf{e} \mapsto \partial$, $\mathsf{h} \mapsto -2t\partial$, $\mathsf{f} \mapsto -t^2\partial$

Lie algebra product $[a, b] = ab - ba$

σ-Twisted vector fields

$\mathsf{e} \mapsto \partial_\sigma$, $\mathsf{h} \mapsto -2t\partial_\sigma$, $\mathsf{f} \mapsto -t^2\partial_\sigma$

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Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$

$$\begin{aligned}\langle a\cdot\Delta, b\cdot\Delta\rangle_\sigma &= (\sigma(a)\cdot\Delta)(b\cdot\Delta)-(\sigma(b)\cdot\Delta)(a\cdot\Delta)\\ &= (\sigma(a)\Delta(b)-\sigma(b)\Delta(a))\cdot\Delta\end{aligned}$$

Assumption $\sigma(1)=1$ and $\partial_\sigma(1)=0$

$$\Downarrow$$

$$\begin{aligned}\langle \mathrm{h},\mathrm{f}\rangle &= 2\sigma(t)\partial_\sigma(t)t\partial_\sigma\\ \langle \mathrm{h},\mathrm{e}\rangle &= 2\partial_\sigma(t)\partial_\sigma\\ \langle \mathrm{e},\mathrm{f}\rangle &= -(\sigma(t)+t)\partial_\sigma(t)\partial_\sigma\end{aligned}$$

Closure of the bracket on $L=\mathbb{K}\mathrm{e}\bigoplus\mathbb{K}\mathrm{f}\bigoplus\mathbb{K}\mathrm{h}$

$$\Updownarrow$$

$$\deg\sigma(t)\partial_\sigma(t)t\leq 2$$

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$. Affine $\sigma(t)$ and ∂_σ is $c\frac{d}{dx}$-like on t^k

Case 1: $\mathcal{A} = \mathbb{K}[t]$, $\sigma(t) = q_0 + q_1 t$, $\partial_\sigma(t) = p_0$

$$\langle \mathrm{h}, \mathrm{f} \rangle : \ -2q_0\mathrm{ef} + q_1\mathrm{hf} + q_0^2\mathrm{eh} - q_0 q_1 \mathrm{h}^2 - q_1^2\mathrm{fh} = -q_0 p_0 \mathrm{h} - 2q_1 p_0 \mathrm{f}$$

$$\langle \mathrm{h}, \mathrm{e} \rangle : \ -2q_0\mathrm{e}^2 + q_1\mathrm{he} - \mathrm{eh} = 2p_0\mathrm{e}$$

$$\langle \mathrm{e}, \mathrm{f} \rangle : \ \mathrm{ef} + q_0^2\mathrm{e}^2 - q_0 q_1\mathrm{he} - q_1^2\mathrm{fe} = -q_0 p_0 \mathrm{e} + \frac{q_1+1}{2} p_0 \mathrm{h}.$$

$q_1 = 1$, $q_0 = p_0 = 0$ gives $\mathfrak{sl}_2(\mathbb{K})$

${}_q\mathfrak{sl}_2(\mathbb{K})$ Jackson $\mathfrak{sl}_2(\mathbb{K})$ (quasi-Lie algebra). Linear $\sigma(t)$ and ∂_σ is $c\frac{d}{dx}$-like on t^k

$q_0 = 0,\ q = q_1 \neq 0\ (\frac{d}{dt} \mapsto D_q)$

$$\begin{aligned} \mathrm{hf} - q\mathrm{fh} &= -2p_0\mathrm{f} \\ \mathrm{he} - q^{-1}\mathrm{eh} &= 2q^{-1}p_0\mathrm{e} \\ \mathrm{ef} - q^2\mathrm{fe} &= \frac{q+1}{2}p_0\mathrm{h} \end{aligned}$$

Iterated Ore extension of $\mathbb{K}[z]$, Auslander-regular, global dimension at most three, has PBW-basis, noetherian domain of GK-dimension three, Koszul as an almost quadratic algebra

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Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$

$p_0 = 0 \to$ "abelianized" version. But $\partial_\sigma = 0$

$p_0 = 1$:

$\langle \mathrm{h}, \mathrm{f} \rangle = -2q\mathrm{f}$, $\langle \mathrm{h}, \mathrm{e} \rangle = 2\mathrm{e}$, $\langle \mathrm{e}, \mathrm{f} \rangle = \frac{q+1}{2}\mathrm{h}$

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$. Twisted (Hom-Lie) Jacobi identity

$$\alpha(\mathsf{e}) = \frac{q^{-1}+1}{2}\,\mathsf{e}, \quad \alpha(\mathsf{h}) = \mathsf{h}, \quad \alpha(\mathsf{f}) = \frac{q+1}{2}\,\mathsf{f}$$

$$\langle \alpha(\mathsf{e}), \langle \mathsf{f}, \mathsf{h} \rangle\rangle + \langle \alpha(\mathsf{f}), \langle \mathsf{h}, \mathsf{e} \rangle\rangle + \langle \alpha(\mathsf{h}), \langle \mathsf{e}, \mathsf{f} \rangle\rangle = 0$$

Quasi-Lie deformations of $\mathfrak{sl}_2(\mathbb{K})$ on $\mathbb{K}[t]/(t^3)$

$\mathcal{A} = \mathbb{K}[t]/(t^3)$, $\sigma(t) = q_1 t + q_2 t^2$, $\partial_\sigma(t) = p_1 t$.

$$\begin{aligned}
\langle h, f\rangle : &\quad q_1 hf + 2q_2 f^2 - q_1^2 fh = 0 \\
\langle h, e\rangle : &\quad q_1 he + 2q_2 fe - eh = -p_1 h - 2p_2 f \\
\langle e, f\rangle : &\quad ef - q_1^2 fe = p_1(q_1 + 1)f.
\end{aligned}$$

$\mathfrak{sl}_2(\mathbb{K})$ cannot be recovered in any "limit"
$p_1 = 0$ representation collapse $\partial_\sigma(t) = 0$

Quasi-Lie quasi-deformations of $\mathfrak{sl}_2(\mathbb{K})$ on $\mathbb{K}[t]/(t^3)$. Special limits

"non-commutative deformation of $\mathbb{K}[x,y]$ in f-direction"

$q_1 = 1, q_2 = -\frac{1}{2}, p_1 = p_2 = 0$

$\mathsf{hf} - \mathsf{fh} = \mathsf{f}^2, \quad \mathsf{he} - \mathsf{eh} = \mathsf{fe}, \quad \mathsf{ef} - \mathsf{fe} = 0$

$\mathsf{f} = 0 \to \mathbb{K}[h, e]$

Quasi-Lie quasi-deformations of $\mathfrak{sl}_2(\mathbb{K})$ on $\mathbb{K}[t]/(t^3)$. Special limits.

solvable 3-dimensional Lie algebra

$q_1 = 1,\ q_2 = 0,\ p_1 = 1,\ p_2 = a/2$

$\mathrm{hf} - \mathrm{fh} = 0, \mathrm{he} - \mathrm{eh} = -\mathrm{h} - a\mathrm{f}, \mathrm{ef} - \mathrm{fe} = 2\mathrm{f}$

Heisenberg Lie algebra

$p_1 = 0, p_2 = -1/2$

$\mathrm{hf} - \mathrm{fh} = 0, \mathrm{he} - \mathrm{eh} = \mathrm{f}, \mathrm{ef} - \mathrm{fe} = 0$

Polynomials in 3 commuting variables

$q_1 = 1,\ q_2 = 0,\ p_1 = p_2 = 0 \to \mathbb{K}[x, y, z]$

Quasi-Hom-Lie algebra Jacobi identity. Case $q_1 p_1 \neq 0$

$$\circlearrowleft_{x,y,z} \Big(\langle\sigma(x),\langle y,z\rangle\rangle + \underbrace{(1 - \frac{q_1 p_2 - p_2 - p_1 q_2}{p_1} t + \xi_2 t^2)}_{=\delta} \langle x,\langle y,z\rangle\rangle\Big) = 0$$

Quasi-Lie deformations on the algebra $\mathbb{K}[t]/(t^N)$

$\mathbb{K}$ include all N^{th}-roots of unity
$\mathcal{A} = \mathbb{K}[t]/(t^N)$ for $N \geq 2$ N-dimensional $\mathbb{K}$-vector space and a finitely generated $\mathbb{K}[t]$-module with basis $\{1, t, \ldots, t^{N-1}\}$.

Quasi-Lie deformations on the algebra $\mathbb{K}[t]/(t^N)$

$t^N = 0$ in the ring $\mathbb{K}[t]/(t^N)$

$$\partial_\sigma(t) = p(t) = \sum_{k=0}^{N-1} p_k t^k, \quad \sigma(t) = \sum_{k=0}^{N-1} q_k t^k$$

Quasi-Lie deformations on the algebra $\mathbb{K}[t]/(t^N)$

Commutation relations

$g_i = c_i t^i \partial_\sigma, \; c_i \in \mathbb{K}, \; c_i \neq 0.$

The bracket is closed on linear span of g_i's For $N-1 \geq i, j \geq 0$

$$\langle g_i, g_j \rangle = c_i c_j [\partial_\sigma(t^j)\sigma(t)^i - \sigma(t)^j \partial_\sigma(t^i)]\partial_\sigma$$

$$= c_i c_j \sum_{k=0}^{|j-i|-1} \operatorname{sign}(j-i) \sum_{\substack{k_1, k_2, \ldots, k_{N-1} \geq 0 \\ k_1 + k_2 + \ldots + k_{N-1} = k + \min\{i,j\} \\ k_2 + 2k_3 + \ldots + (N-2)k_{N-1} < N}} \frac{(k + \min\{i,j\})!}{k_1! k_2! \cdots k_{N-1}!}$$

$$\times \, q_1^{k_1} q_2^{k_2} \ldots q_{N-1}^{k_{N-1}} t^{k_2 + 2k_3 + \ldots + (N-2)k_{N-1}} \sum_{l=0}^{N-1} p_l t^{i+j+l-1} \partial_\sigma$$

Quasi-Lie deformations on the algebra $\mathbb{K}[t]/(t^N)$

$$= c_i c_j \sum_{l=0}^{N-1} p_l \sum_{k=0}^{|j-i|-1} \operatorname{sign}(j-i) \sum_{\substack{k_1,k_2,\ldots,k_{N-1}\geq 0 \\ k_1+k_2+\ldots+k_{N-1}=k+\min\{i,j\} \\ k_2+2k_3+\ldots+(N-2)k_{N-1}\leq N-i-j-l}} \frac{(k+\min\{i,j\})!}{k_1!k_2!\cdots k_{N-1}!}$$

$$\times q_1^{k_1} q_2^{k_2} \ldots q_{N-1}^{k_{N-1}} \frac{g_{i+j+l-1+k_2+2k_3+\ldots+(N-2)k_{N-1}}}{c_{i+j+l-1+k_2+2k_3+\ldots+(N-2)k_{N-1}}}$$

where $\operatorname{sign}(x) = -1$ if $x < 0$, $\operatorname{sign}(x) = 0$ if $x = 0$ and $\operatorname{sign}(x) = 1$ if $x > 0$.

Hom-associative algebras $\mapsto$ Hom-Lie algebras

Hom-associative algebras were first introduced in

Makhlouf A., Silvestrov S.D., Hom-algebra structures, J. Gen. Lie Theory, Appl. 2 (2), 51–64 (2008).

(Preprints in Mathematical Sciences 2006:10, LUTFMA-5074-2006, Centre for Mathematical Sciences,

Department of Mathematics, Lund Institute of Technology, Lund University, 2006)

Hom-associative algebra (V, μ, α)

V linear space,

$\mu : V \times V \to V$ bilinear map,

$\alpha : V \to V$ linear map (linear space endomorphism of V)

Hom-associativity (two notations for multiplication)

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))$$

$$\alpha(x)(yz) = (xy)\alpha(z)$$

$\alpha = Id_V \Leftrightarrow$ associative algebra

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Hom-associative algebras $\mapsto$ Hom-Lie algebras

Theorem

Hom-associative algebras are **Hom-Lie admissible**:
For any Hom-associative algebra (V, μ, α),
$(V, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra
with commutator bracket multiplication

$$[x, y] = \mu(x, y) - \mu(y, x)$$

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

G subgroup of the permutations group $\mathcal{S}_3$

Definition Hom-algebra (V, μ, α) is said to be G-**Hom-associative** if

$$\forall x_1, x_2, x_3 \in V$$

$$\sum_{s \in G = \mathcal{S}_3} (-1)^{\varepsilon(s)} \underbrace{(\mu(\mu(x_{s(1)}, x_{s(2)}), \alpha(x_{s(3)})) - \mu(\alpha(x_{s(1)}), \mu(x_{s(2)}, x_{s(3)})))}_{a_{\mu,\alpha}} = 0,$$

Equivalently

$$\sum_{s \in G} (-1)^{\varepsilon(s)} a_{\mu,\alpha} \circ \sigma = 0$$

$s(x_1, x_2, x_3) = (x_{s(1)}, x_{s(2)}, x_{s(3)})$,

$(-1)^{\varepsilon(s)}$ is the signature of the permutation s.

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

Theorem

G-Hom-associative algebras are Hom-Lie admissible:
For any G-Hom-associative algebra (V, μ, α),
$(V, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra
with commutator bracket multiplication

$$[x, y] = \mu(x, y) - \mu(y, x)$$

G-Hom-associative algebras

The subgroups of $\mathcal{S}_3$ are

$G_1 = \{Id\}, \quad G_2 = \{Id, \tau_{12}\}, \quad G_3 = \{Id, \tau_{23}\}$

$G_4 = \{Id, \tau_{13}\}, \quad G_5 = A_3, \quad G_6 = \mathcal{S}_3$

A_3 is the alternating group;

τ_{ij} is the transposition of i and j.

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

- The G_1-Hom-associative algebras are the Hom-associative algebras.
- The G_2-Hom-associative algebras satisfy

$$\mu(\alpha(x),\mu(y,z))-\mu(\alpha(y),\mu(x,z))=\mu(\mu(x,y),\alpha(z))-\mu(\mu(y,x),\alpha(z))$$

Vinberg algebra or left symmetric algebra: $\alpha = Id$

- The G_3-Hom-associative algebras satisfy

$$\mu(\alpha(x),\mu(y,z))-\mu(\alpha(x),\mu(z,y))=\mu(\mu(x,y),\alpha(z))-\mu(\mu(x,z),\alpha(y))$$

Pre-Lie algebra or right symmetric algebra: $\alpha = Id$

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

- The G_4-Hom-associative algebras satisfy

$$\mu(\alpha(x),\mu(y,z)) - \mu(\alpha(z),\mu(y,x)) = \mu(\mu(x,y),\alpha(z)) - \mu(\mu(z,y),\alpha(x))$$

- The G_5-Hom-associative algebras satisfy the condition

$$\mu(\alpha(x),\mu(y,z)) + \mu(\alpha(y),\mu(z,x)) + \mu(\alpha(z),\mu(x,y)) = \mu(\mu(x,y),\alpha(z)) + \mu(\mu(y,z),\alpha(x)) + \mu(\mu(z,x),\alpha(y))$$

  Note: μ skew-symmetric $\Rightarrow$ the Hom-Jacobi identity.

- The G_6-Hom-associative algebras are the Hom-Lie admissible algebras.

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

A Hom-pre-Lie algebra is a triple (V, μ, α) consisting of a linear space V, a bilinear map $\mu : V \times V \to V$ and a homomorphism α satisfying

$$\mu(\alpha(x), \mu(y, z)) - \mu(\alpha(x), \mu(z, y)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(x, z), \alpha(y))$$

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

Theorem

G-Hom-associative algebras are Hom-Lie admissible:
For any G-Hom-associative algebra (V, μ, α),
$(V, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra
with commutator bracket multiplication

$$[x, y] = \mu(x, y) - \mu(y, x)$$

From Lie algebras to Hom-Lie algebras. Composition trick

$(V, [\cdot,\cdot])$ Lie algebra
$\alpha : V \to V$ Lie algebra endomorphism

$$[x, y]_\alpha = \alpha([x, y])$$

Then $(V, [\cdot,\cdot]_\alpha)$ is a Hom-Lie algebra

$$[x, y]_\alpha = -[y, x]_\alpha, \qquad \circlearrowleft_{x,y,z} [[\alpha(x), [y, z]_\alpha]_\alpha = 0.$$

Hom-Leibniz algebras (Hom-Loday algebras)

Special subclass of quasi-Leibniz algebras

Quasi-Leibniz algebras were first introduced in

D. Larsson, S. Silvestrov, Quasi-Lie algebras, In: Jurgen Fuchs, et al. (eds), "Noncommutative Geometry and Representation Theory in Mathematical Physics", American Mathematical Society, Contemporary Mathematics, Vol. 391, 2005.

Definition $(V, \langle\cdot,\cdot\rangle, \alpha)$ consisting of a linear space V, bilinear map $\langle\cdot,\cdot\rangle : V \times V \to V$ and a homomorphism $\alpha : V \to V$ satisfying

$$\langle\langle x, y\rangle, \alpha(z)\rangle = \langle\langle x, z\rangle, \alpha(y)\rangle + \langle\alpha(x), \langle y, z\rangle\rangle.$$

If a Hom-Leibniz algebra is skewsymmetric then it is a Hom-Lie algebra.

Hom-Poisson algebra

Definition

$(V, \mu, \{\cdot,\cdot\}, \alpha)$

V linear space, $\mu : V \times V \to V$ and $\{\cdot,\cdot\} : V \times V \to V$ bilinear maps

$\alpha : V \to V$ linear map:

1) (V, μ, α) is a commutative Hom-associative algebra

2) $(V, \{\cdot,\cdot\}, \alpha)$ is a Hom-Lie algebra

3) for all x, y, z in V,

$$\{\alpha(x), \mu(y,z)\} = \mu(\alpha(y), \{x,z\}) + \mu(\alpha(z), \{x,y\}).$$

Hom-Poisson algebra

Equivalently:

$$\{\mu(x,y),\alpha(z)\} = \mu(\{x,z\},\alpha(y)) + \mu(\alpha(x),\{y,z\})$$

for all x, y, z in V.
$ad_z(\cdot) = \{\cdot, z\}$ is a Hom-derivation for the multiplication μ

Hom-Poisson algebra

Let $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ be a deformation of the commutative Hom-associative algebra
$\mathcal{A}_0 = (V, \mu_0, \alpha_0)$

$$\mu_t(x,y) = \mu_0(x,y) + \mu_1(x,y)t + \mu_2(x,y)t^2 + \dots$$

Then

$$\frac{\mu_t(x,y) - \mu_t(y,x)}{t} = \mu_1(x,y) - \mu_1(y,x) + t\sum_{i\geq 2}(\mu_i(x,y) - \mu_i(y,x))t^{i-2}$$

Hence, if t goes to zero then $\frac{\mu_t(x,y)-\mu_t(y,x)}{t}$ goes to
$\{x,y\} := \mu_1(x,y) - \mu_1(y,x)$

Hom-Poisson algebra

Theorem
$\mathcal{A}_0 = (V, \mu_0, \alpha_0)$
a commutative Hom-associative algebra
$\mathcal{A}_t = (V, \mu_t, \alpha_t)$ a deformation of $\mathcal{A}_0$.
Consider the bracket
$\{x, y\} = \mu_1(x, y) - \mu_1(y, x)$
is the first order element of the deformation μ_t.

$$\Downarrow$$

$(V, \mu_0, \{,\}, \alpha_0)$ is a Hom-Poisson algebra.

Hom-Nambu and Hom-Nambu-Lie n-ary algebras

H. Ataguema, A. Makhlouf, S. Silvestrov, Generalization of n-ary Nambu Algebras and Beyond, Journal of Mathematical Physics, 50, 083501,2009

Definition

An *n-ary Hom-Nambu algebra* is a triple $(V, [\cdot, \dots, \cdot], \alpha)$, where V is linear space,
$\alpha = (\alpha_i)_{i=1,\cdots,n-1}$ is a family of linear maps $\alpha_i : V \to V$,
$[\cdot, \cdots, \cdot] : V^{\times n} \to V$ is n-linear map (n-ary product) satisfying:
The n-ary Hom-Nambu identity

$$[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [x_n, \dots, x_{2n-1}]] =$$
$$\sum_{i=n}^{2n-1} [\alpha_1(x_n), \dots, \alpha_{i-n}(x_{i-1}), [x_1, \cdots, x_{n-1}, x_i], \alpha_{i-n+1}(x_{i+1}) \dots, \alpha_{n-1}(x_{2n-1})]$$

for all $(x_1, \cdots, x_{2n-1}) \in V^{2n-1}$.

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n-ary Hom-Nambu and Hom-Nambu-Lie algebras

Ternary Hom-Nambu algebras

$$[\alpha_1(x_1),\alpha_2(x_2),[x_3,x_4,x_5]] = [[x_1,x_2,x_3],\alpha_1(x_4),\alpha_2(x_5)] + [\alpha_1(x_3),[x_1,x_2,x_4],\alpha_2(x_5)] + [\alpha_1(x_3),\alpha_2(x_4),[x_1,x_2,x_5]].$$

n-ary Hom-Nambu algebras

Theorem. Let (V, m) be an n-ary Nambu algebra and let $\rho : V \to V$ be an n-ary Nambu algebras endomorphism.
$m_\rho = \rho \circ m$
$\widetilde{\rho} = (\rho, \cdots, \rho)$.
Then $(V, m_\rho, \widetilde{\rho})$ is an n-ary Hom-Nambu algebra.

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n-ary Hom-Nambu-Lie algebras

Definition

A ternary Hom-Nambu algebra $\big(V, [\cdot,\cdot,\cdot], (\alpha_1, \alpha_2)\big)$ is called a *ternary Hom-Nambu-Lie algebra* if the bracket is skew-symmetric, that is

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = Sgn(\sigma)[x_1, x_2, x_3]$$

$\forall\, \sigma \in \mathcal{S}_3$ and $\forall\, x_1, x_2, x_3 \in V$

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

J. Arnlind, N. Makhlouf, S. Silvestrov, Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras, Journal of Mathematical Physics 51, 1, 2010

Definition

$(V, [\cdot,\cdot])$ binary algebra
$\tau : V \to \mathbb{K}$ linear map.
Define ternary bracket (trilinear map)
$[\cdot,\cdot,\cdot]_\tau : V \times V \times V \to V$:

$$[x, y, z]_\tau = \tau(x)[y, z] + \tau(y)[z, x] + \tau(z)[x, y].$$

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

If the bilinear multiplication $[\cdot,\cdot]$ in Definition 3 is skew-symmetric, then the trilinear map $[\cdot,\cdot,\cdot]_\tau$ is skew-symmetric as well.

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Hom-Nambu-Lie algebras induced from Hom-Lie algebras

If τ is a linear function such that $\tau([x, y]) = 0$ for all $x, y \in V$, then we call τ a *trace function on* $(V, [\cdot, \cdot])$. It follows immediately that $\tau([x, y, z]_\tau) = 0$ for all $x, y, z \in V$ if τ is a trace function.

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Theorem

$(V, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra and $\beta : V \to \mathbb{K}$ be a linear map. Assume that τ is a trace function on V fulfilling

$$\tau\big(\alpha(x)\big)\tau(y) = \tau(x)\tau\big(\alpha(y)\big)$$
$$\tau\big(\beta(x)\big)\tau(y) = \tau(x)\tau\big(\beta(y)\big)$$
$$\tau\big(\alpha(x)\big)\beta(y) = \tau\big(\beta(x)\big)\alpha(y)$$

for all $x, y \in V$.
Then $(V, [\cdot,\cdot,\cdot]_\tau, (\alpha, \beta))$ is a Hom-Nambu-Lie algebra.

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

If we choose $\beta = \alpha$ conditions for trace τ reduce to

$$\tau(\alpha(x))\tau(y) = \tau(x)\tau(\alpha(y)).$$

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Example

V vector space of $n \times n$ matrices
$\alpha(x) = s^{-1}xs$ for an invertible matrix s
Then $(V, \alpha \circ [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra. For matrices, any trace function is proportional to the matrix trace, so we let $\tau(x) = \mathrm{tr}(x)$. If we want to choose a $\beta \neq 0$, it can be proved that β has to be proportional to α, i.e. $\beta = \lambda\alpha$ for some $\lambda \neq 0$. Since $\mathrm{tr}\left(\alpha(x)\right) = \mathrm{tr}(x)$ it is clear that $(\alpha, \lambda\alpha, \mathrm{tr})$ is a nondegenerate compatible triple on V, which implies that $\left(V, [\cdot,\cdot,\cdot]_{\mathrm{tr}}, (\alpha, \lambda\alpha)\right)$ is a Hom-Nambu-Lie algebra induced from $(V, \alpha \circ [\cdot,\cdot], \alpha)$.

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Example

Let us start with the vector space V spanned by $\{x_1, x_2, x_3, x_4\}$ with a skew-symmetric bilinear map defined through

$$[x_i, x_j] = a_{ij}x_3 + b_{ij}x_4$$

where a_{ij} and b_{ij} are antisymmetric 4×4 matrices. Defining

$$\begin{aligned} &\alpha(x_i) = x_3 \qquad \beta(x_i) = x_4 \qquad i = 1, \dots, 4 \\ &\tau(x_1) = \gamma_1 \qquad \tau(x_2) = \gamma_2 \qquad \tau(x_3) = \tau(x_4) = 0, \end{aligned}$$

one immediately observes that τ is a trace function, $\operatorname{im}\alpha \subseteq \ker\tau$, $\operatorname{im}\beta \subseteq \ker\tau$, and $\beta \neq \alpha$.

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Example cont.

Furthermore, $(V, [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra provided

$$b_{13} = b_{12} + b_{23}$$
$$b_{14} = b_{12} + b_{23} + b_{34}$$
$$b_{24} = b_{23} + b_{34}.$$

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Example cont.
By introducing $a = b_{12}$, $b = b_{23}$ and $c = b_{34}$, the four independent ternary brackets of the induced Hom-Nambu-Lie algebra can be written as

$$[x_1, x_2, x_3] = (\gamma_1 a_{23} - \gamma_2 a_{13})x_3 + (\gamma_1 b - \gamma_2(a + b))x_4$$
$$[x_1, x_2, x_4] = (\gamma_1 a_{24} - \gamma_2 a_{14})x_3 + (\gamma_1(b + c) - \gamma_2(a + b + c))x_4$$
$$[x_1, x_3, x_4] = (\gamma_1 a_{34})x_3 + (\gamma_1 c)x_4$$
$$[x_2, x_3, x_4] = (\gamma_2 a_{34})x_3 + (\gamma_2 c)x_4.$$

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Example cont.
For instance, choosing $\gamma_1 = \gamma_2 = 1$ and $a_{i<j} = 1$, one obtains the Hom-Nambu-Lie algebra

$$(\langle x_1, x_2, x_3, x_4\rangle, [\cdot,\cdot,\cdot], (\alpha,\beta))$$

defined by

$$\begin{aligned}
[x_1, x_2, x_3] &= -a x_4\\
[x_1, x_2, x_4] &= -c x_4\\
[x_1, x_3, x_4] &= x_3 + c x_4\\
[x_2, x_3, x_4] &= x_3 + c x_4
\end{aligned}$$

together with $\alpha(x_i) = x_3$ and $\beta(x_i) = x_4$.

Some key references on Hom-algebra structures

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Thank you for the music,

the songs I'm singing!

Thanks for all the joy

they're bringing!