Separability of object unital groupoid graded rings

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Outline

- Separable field extensions
- Separable unital ring extensions
- Separability for strongly group graded rings
- Separable functors
- Separable nonunital ring extensions
- Separability for object unital strongly groupoid graded rings

Definition

(Galois, Weber 1893, Steinitz 1910)

An algebraic field extension L/K is called separable if for all $a \in L$ the minimal polynomial of a does not have any double roots in an algebraic closure of K.

Theorem A

If L/K is a finite field extension, then the following are equivalent:

- **L/K is separable.**
- **$1 \in tr(L)$.**

Definition

Let L/K be a finite field extension of degree n. Given $f \in End_K(L)$ define $Tr(f)$ as the diagonal sum of a matrix representation of f in $M_n(K)$. Given $a \in L$ define $\lambda_a \in End_K(L)$ as the map

$$\lambda_a(x) = ax$$

for $x \in L$. Define the field trace map

$$tr : L \to K$$

by

$$tr(a) = Tr(\lambda_a)$$

for $a \in L$.

Tensor products

($\mathbb{R}^n$: Gibbs 1886, Abelian groups: Whitney 1938, Modules: Bourbaki 1948)

If

$$L_1/K$$

and

$$L_2/K$$

are field extensions, then

$$L_1 \otimes_K L_2$$

is a commutative K-algebra which may not be a field. Hence, it may contain nilpotent elements.

Definition

(Köthe 1930)

If A is a commutative finite dimensional K-algebra, then the nilradical $Nil(A)$ of A is the ideal of A consisting of the set of nilpotent elements of A.

Proposition

If A is a commutative finite dimensional K-algebra, then $Nil(A) = \{0\} \Leftrightarrow A$ is a finite direct product of fields. In that case, A is a semisimple algebra and hence all left A-modules are projective.

Proposition

If L/K is a finite field extension, then the following are equivalent:

- L/K is separable.

- $Nil(L \otimes_K L) = \{0\}$.

- L is projective as an $L \otimes_K L$-module (or equivalently as an L-bimodule).

- The multiplication map $m : L \otimes_K L \to L$ is split by some L-bimodule homomorphism $\delta : L \to L \otimes_K L$.

- There exists $e \in L \otimes_K L$ with $m(e) = 1$ and $el = le$ for all $l \in L$ (choose $e = \delta(1)$ from above). Such an element e is called a separability idempotent.

Definition

**(B commutative: Auslander-Goldman 1960,
B noncommutative: Hirata-Sugano 1966)**

Suppose that A/B is a ring extension of unital rings with the same 1. Then A/B is called separable if either of the following equivalent conditions hold:

- A is projective as an $A \otimes_B A^{\text{op}}$-module (or equivalently as an A-bimodule).

- The multiplication map $m : A \otimes_B A \to A$ is split by some A-bimodule homomorphism $\delta : A \to A \otimes_B A$.

- There exists $e \in A \otimes_B A$ with $m(e) = 1$ and $ea = ae$ for all $a \in A$ (choose $e = \delta(1)$ above).

Examples

- **Group ring $A = B[G]$ with B unital. Then A/B is separable if and only if G is finite and $|G|$ is invertible in B. In that case:**

$$e = |G|^{-1} \sum_{g \in G} g \otimes g^{-1}.$$

- **Matrix ring $A = M_n(B)$ with B unital. Then A/B is always separable with:**

$$e = \sum_{i=1}^{n} e_{ij} \otimes e_{ji}$$

for any fixed j.

Definition

Let R be a ring and G a group with identity e. The ring R is said to be graded by G, or G-graded, if for all $g \in G$ there is an additive subgroup R_g of R such that:

- **$R = \oplus_{g \in G} R_g$ as additive groups, and**
- **for all $g, h \in G$ the inclusion $R_g R_h \subseteq R_{gh}$ holds.**

In that case R is said to be strongly graded by G if for all $g, h \in G$ the equality $R_g R_h = R_{gh}$ holds.

Theorem B

(Miyashita 1970, Nastasescu-Van Oystaeyen 1989)

If R is a unital strongly G-graded ring, then the following are equivalent:

- R/R_e **is separable**
- $1 \in tr(Z(R_e))$

Remark

Theorem B has been extended to epsilon-strongly group graded rings by •, Öinert and Pinedo (2018).

Definition

Let R be a unital strongly G-graded ring. For all $g \in G$ take $n_g \in \mathbb{N}$ and $u_g^{(i)} \in R_g$ and $v_{g^{-1}}^{(i)} \in R_{g^{-1}}$, for $i = 1, \ldots, n_g$, such that:

$$1 = \sum_{i=1}^{n_g} u_g^{(i)} v_{g^{-1}}^{(i)} \qquad (1 \in R_g R_{g^{-1}})$$

For $g \in G$ define $\gamma_g : Z(R_e) \to Z(R_e)$ by:

$$\gamma_g(x) = \sum_{i=1}^{n_g} u_g^{(i)} x v_{g^{-1}}^{(i)}$$

for $x \in Z(R_e)$. If G is finite, then put:

$$tr(x) = \sum_{g \in G} \gamma_g(x)$$

for $x \in Z(R_e)$.

Proposition

(B commutative: Hattori 1963,
B noncommutative: Hirata-Sugano 1966)

If A/B be a separable extension of unital rings, then every submodule of a left A-module which is a B-direct summand is an A-direct summand (in other words, A/B is "semisimple").

Definition

Let A/B be a ring extension. We let:

$$\varphi : B \to A$$

denote the inclusion map. To φ we associate the restriction functor:

$$\varphi_* : A\text{-}\mathbf{mod} \to B\text{-}\mathbf{mod}$$

which to a left A-module associates its natural structure as a left B-module.

Definition

(Nastasescu-Van Oystaeyen 1989)

Let C **and** D **be categories. A functor** $F : C \to D$ **is called separable if for all** $M, N \in \mathrm{Ob}(C)$**, there is** $\psi_{M,N} : \mathrm{Hom}_D(F(M), F(N)) \to \mathrm{Hom}_C(M, N)$ **with:**

- $\psi_{M,M'}(F(\alpha)) = \alpha$

- $F(\beta)f = gF(\alpha) \Rightarrow \beta\psi_{M,N}(f) = \psi_{M',N'}(g)\alpha$

for all $M, N, M', N' \in \mathrm{ob}(C)$**, all** $\alpha \in \mathrm{Hom}_C(M, M')$**, all** $\beta \in \mathrm{Hom}_C(N, N')$**, all** $f \in \mathrm{Hom}_D(F(M), F(N))$ **and all** $g \in \mathrm{Hom}_D(F(N'), F(M'))$**.**

Theorem C
(Nastasescu-Van Oystaeyen 1989)

If A/B is a ring extension of unital rings, then the following are equivalent:

- **A/B is separable.**

- **The restriction functor $\varphi_* : A$-mod $\to B$-mod is separable.**

Remark

Suppose that A/B is a separable extension of unital rings. Let $\sum_{i\in I} x_i \otimes y_i \in A \otimes_B A$ be a separability idempotent for A/B. Define:

$$\psi_{M,N} : \mathrm{Hom}_B(M,N) \to \mathrm{Hom}_A(M,N)$$

by:

$$\psi_{M,N}(f)(m) = \sum_{i\in I} x_i f(y_i m)$$

for $f \in \mathrm{Hom}_B(M,N)$ and $m \in M$.

Proposition

(Nastasescu-Van Oystaeyen 1989)

Let $F : C \to D$ and $G : D \to E$ be functors.

- **F and G separable $\Rightarrow$ GF separable.**
- **GF separable $\Rightarrow$ F separable.**

Suppose that F is separable:

- **$f \in Hom_C(M, N)$ and $F(f)$ has a left (or right, or two-sided) inverse in D $\Rightarrow$ f has a left (or right, or two-sided) inverse in C.**
- **F preserves epimorphisms (monomorphisms) $\Rightarrow$ F reflects projective (injective) objects.**

Definition

By a groupoid $\mathcal{G}$ we mean a small category with the property that all morphisms are isomorphisms. Equivalently, it can be defined by saying that $\mathcal{G}$ is a non-empty set equipped with a unary operation

$$\mathcal{G} \ni \sigma \mapsto \sigma^{-1} \in \mathcal{G} \quad \textbf{(inversion)}$$

and a partial binary operation

$$\mathcal{G} \times \mathcal{G} \ni (\sigma, \tau) \mapsto \sigma\tau \in \mathcal{G} \quad \textbf{(composition)}$$

such that $\forall \sigma, \tau, \rho \in \mathcal{G}$ the following four axioms hold:

- $(\sigma^{-1})^{-1} = \sigma$

- **if** $\sigma\tau$ **and** $\tau\rho$ **are defined, then** $(\sigma\tau)\rho$ **and** $\sigma(\tau\rho)$ **are defined and equal**

- **the domain** $d(\sigma) := \sigma^{-1}\sigma$ **is always defined and if** $\sigma\tau$ **is defined, then** $d(\sigma)\tau = \tau$

- **the range** $r(\tau) := \tau\tau^{-1}$ **is always defined and if** $\sigma\tau$ **is defined, then** $\sigma r(\tau) = \sigma$**.**

Definition

Let $\mathcal{G}$ be a groupoid.

- $\mathcal{G}_0 := d(\mathcal{G}) = r(\mathcal{G})$ **is called the unit space of** $\mathcal{G}$
- $\mathcal{G}_1 := \mathcal{G}$
- $\mathcal{G}_2 := \{(\sigma, \tau) \in \mathcal{G} \times \mathcal{G} \mid \sigma\tau \text{ is defined}\}$

Definition

Suppose that R is a ring and that $\mathcal{G}$ is a groupoid. We say that R is graded by $\mathcal{G}$ if there for all $\sigma \in \mathcal{G}$ is an additive subgroup R_σ of R such that

$$R = \oplus_{\sigma \in \mathcal{G}} R_\sigma$$

and $\forall \sigma, \tau \in \mathcal{G}$

$$R_\sigma R_\tau \subseteq R_{\sigma\tau} \quad \text{if} \quad (\sigma, \tau) \in \mathcal{G}_2$$

$$R_\sigma R_\tau = \{0\} \quad \text{if} \quad (\sigma, \tau) \notin \mathcal{G}_2.$$

In that case, if $\forall (\sigma, \tau) \in G_2 \quad R_\sigma R_\tau = R_{\sigma\tau}$, then we say that R is strongly graded by $\mathcal{G}$.

Definition

If R is a ring which is graded by a groupoid $\mathcal{G}$, then we say that R is object unital if

- $\forall e \in \mathcal{G}_0 \quad R_e$ **is unital, and**
- $\forall \sigma \in \mathcal{G} \quad \forall r \in R_\sigma \quad 1_{R_{r(\sigma)}} r = r 1_{R_{d(\sigma)}} = r$.

Remark

If R is a ring which is graded by a groupoid $\mathcal{G}$, and R is object unital, then R is a ring with enough idempotents (namely the 1_{R_e} for $e \in \mathcal{G}_0$).

Definition

Let $\mathcal{G}$ be a groupoid and $e \in \mathcal{G}_0$. Define the group:

$$\mathcal{G}(e) = \{\sigma \in \mathcal{G} \mid d(\sigma) = r(\sigma) = e\}.$$

This is called the principal group at e. Suppose that R is a ring which is graded by $\mathcal{G}$. Let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$. We put:

$$R_{\mathcal{H}} = \oplus_{\sigma \in \mathcal{H}} R_\sigma.$$

Then $R_{\mathcal{H}}$ is graded by $\mathcal{H}$. We also put:

$$R_0 = R_{\mathcal{G}_0}.$$

Theorem D

(Cala, •, Pinedo 2020)

If $\mathcal{G}$ is a groupoid and R is an object unital strongly $\mathcal{G}$-graded ring, then the following are equivalent:

- **The functor $\varphi_* : R\text{-Mod} \to R_0\text{-Mod}$, associated to the inclusion map $\varphi : R_0 \to R$, is separable.**
- **R/R_0 is separable.**
- **For all $e \in \mathcal{G}_0$, $R_{\mathcal{G}(e)}/R_e$ is separable.**
- **For all $e \in \mathcal{G}_0$, $\mathcal{G}(e)$ is finite and $1_{R_e} \in \text{tr}_e(Z(R_e))$.**

Remark: Bagio and Pinedo (2017) have shown a version of Theorem D that holds for partial skew groupoid rings.

Definition
(Taylor 1982, Brzezinski 2002)

Suppose that A and B are (not necessarily unital) rings with $A \supseteq B$. Then A/B is called separable if the multiplication map:

$$m : A \otimes_B A \to A$$

is split by some A-bimodule homomorphism:

$$\delta : A \to A \otimes_B A.$$

Theorem E
(Cala, •, Pinedo 2020)

If A/B is an extension of rings with enough idempotents such that $\{e_i\}_{i\in I} \subseteq B$ is a complete set of idempotents for both A and B, then the following are equivalent:

• The functor $\varphi_* : A$-Mod $\to B$-Mod, associated to the inclusion map $\varphi : B \to A$, is separable

• A/B is separable.

• For all $i \in I$ there exists an element $x_i \in \sum_{j\in I} e_iAe_j \otimes_B e_jAe_i$ such that for all $j \in I$ and all $a \in e_iAe_j$, the equalities $\mu(x_i) = e_i$ and $x_ia = ax_j$ hold.

Questions

- Can we generalize Theorem D to epsilon-strongly groupoid graded rings?

- What is the connection between Theorem D (or at least Theorem B) and Theorem A?

Thank you!