# Centralizers in Skew PBW extensions

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# Outline

# Introduction

Skew PBW (Poincare-Birkoff-Witt) extensions also known as $\sigma-$PBW extensions are a wide class of non commutative rings which include many rings and algebras arising in quantum mechanics and many areas in mathematics, such as the classical PBW extensions, Weyl algebras, enveloping algebras of finite dimensional Lie algebras, iterated Ore extensions of injective type and many others.

## Definition

Let $R$ and $A$ be rings. $A$ is a $\sigma-$PBW extension of $R$ (or skew PBW extension), if the following conditions hold:

(a) $R \subseteq A$.

(b) There exist finite elements $x_1, \cdots, x_n$ such that $A$ is a left $R-$free module with basis

$$Mon(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \ : \ \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n\}.$$

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That is, $A$ is a left polynomial ring over $R$ with respect to $\{x_1, \cdots, x_n\}$ and $Mon(A)$ is the set of standard monomials in $A$. Moreover $x_1^0 \cdots x_n^0 := 1 \in Mon(A)$.

(c) For every $1 \leqslant i \leqslant n$ and $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R.$$

## Definition

(d) For every $1 \leqslant i, j \leqslant n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n.$$

Under these conditions we write $A = \sigma(R)\langle x_1, \cdots, x_n \rangle$.

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**Theorem (C. Gallego, O. Lezama, 2010)**

*Let $A$ be a skew PBW extension of $R$. Then for every $1 \leqslant i \leqslant n$, there exists an injective ring endomorphism $\sigma_i : R \to R$ and a $\sigma_i$−derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r)x + \delta_i(r)$$

*for each $r \in R$.*

## Definition

Let $A$ be a $\sigma-$PBW extension.

1. $A$ is quasi-commutative if the conditions $(c)$ and $(d)$ are replaced by:
   (c') For every $1 \leqslant i \leqslant n$ and $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that
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2. $A$ is bijective if $\sigma_i$ is bijective for every $1 \leqslant i \leqslant n$ and $c_{i,j}$ is invertible for any $1 \leqslant i < j \leqslant n$.

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- Any injective type iterated Ore algebra $\mathbb{K}[t_1, \cdots, t_m][x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$, ($\mathbb{K}$ a field), with $\sigma_i$ injective for all $1 \leq i \leqslant n$ is a skew PBW extension.

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- Any injective type iterated Ore algebra $\mathbb{K}[t_1, \cdots, t_m][x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$, ($\mathbb{K}$ a field), with $\sigma_i$ injective for all $1 \leq i \leqslant n$ is a skew PBW extension. Examples include algebra of shift operators, additive analogue of Weyl algebra, multiplicative analogue of the Weyl algebra, $q-$Heisenberg algebra, etc.

## Definition

- For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$
- For $X = x^\alpha \in Mon(A)$, $exp(X) := \alpha$ and $deg(X) = |\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$.
- Let $0 \neq f \in A$ such that $f = c_1 X_1 + \cdots + c_t X_t$ with $X_i \in Mon(A)$ and $c_i \in R \setminus \{0\}$ then $deg(f) = \max\{deg(X_i)\}_{i=1}^t$.

## Theorem (C. Gallego, O. Lezama, 2010)

*Let $A$ be a left polynomial ring over $R$ w.r.t. $\{x_1, x_2, \cdots, x_n\}$. Then $A$ is a skew PBW extension if and only if the following conditions hold.*

- *For every $x^\alpha \in Mon(A)$ and $0 \neq r \in R$ there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$ and $P_{\alpha,r} \in A$ such that*

$$x^\alpha r = r_\alpha x^\alpha + P_{\alpha,r}$$

  *where $P_{\alpha,r} = 0$ or $deg(P_\alpha, r) < |\alpha|$, if $P_{\alpha,r} \neq 0$.*

- *For every $x^\alpha, x^\beta \in Mon(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $P_{\alpha,\beta} \in A$ such that*

$$x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + P_{\alpha,\beta},$$

  *where $c_{\alpha,\beta}$ is left invertible and $P_{\alpha,\beta} = 0$, or $deg(P_{\alpha,\beta}) < |\alpha + \beta|$ if $P_{\alpha,\beta} \neq 0$*

# Quasi-Commutative Case

## Theorem

*Let $R$ be a commutative ring and suppose that for all $1 \leqslant i \leqslant n$, $\delta_i = 0$. Then the centralizer $C(R)$ of $R$ in the skew PBW extension $\sigma(R) < x_1, \cdots, x_n >$ is given by*

$$C(R) = \left\{ \sum_{\alpha} f_\alpha x^\alpha \ : \ (\forall\ r \in R),\ (\sigma^\alpha(r) - r) f_\alpha = 0 \right\}.$$

# A necessary condition for the general case

**Theorem**

*Let $R$ be a commutative ring. If an element $\sum_\alpha f_\alpha x^\alpha \in \sigma(R) < x_1, \cdots, x_n >$ belongs to the centralizer $C(R)$, then $(\sigma^\alpha(r) - r)f_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$.*

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## Corollary

*Let $R$ be a commutative ring. If for every $\alpha \in \mathbb{N}^n$ there exists $r \in R$ such that $(\sigma^{\alpha}(r) - r)$ is a regular element, then $C(R) = R$.*

Let $\Omega = \{1, \cdots, N\}$ be a finite set and let $\mathbb{R}^{\Omega} = \{f : \Omega \to \mathbb{R}\}$ denote the algebra of real-valued functions on $\Omega$ with respect to the usual pointwise operations.

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$$\tilde{\tau}_i(f) = f \circ \tau_i^{-1} \tag{1}$$

for every $f \in \mathbb{R}^\Omega$ and let $\delta_i, 1 \leqslant i \leqslant n$ be a $\tilde{\tau}_i-$derivation. Consider the skew-PBW extension $\tilde{\tau}\left(\mathbb{R}^\Omega\right) < x_1, \cdots, x_n >$.

## Definition

For $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$, define

(a) $Sep^{\alpha}(\Omega) := \{\omega \in \Omega \ : \ \tau^{\alpha}(\omega) \neq \omega\}$;

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## Theorem

*Suppose that for* $1 \leqslant i \leqslant n$, $\delta_i = 0$. *Then the centralizer* $C\left(\mathbb{R}^{\Omega}\right)$, *of* $\mathbb{R}^{\Omega}$ *in the skew PBW extension* $\tilde{\tau}(\mathbb{R}^{\Omega}) < x_1, \cdots, x_n >$ *is given by*

$$C\left(\mathbb{R}^{\Omega}\right) = \left\{\sum_{\alpha} f_{\alpha} x^{\alpha} \ : \ f_{\alpha} = 0 \text{ on } Sep^{\alpha}(\Omega)\right\}.$$

# Introduction


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# Thank you.