Hom-algebra structures and quasi Hom-Lie algebras

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Outline

1. σ-derivations (twisted or deformed or discretized derivations)
2. Quasi-Hom-Lie algebras of twisted (deformed) vector fields
3. Quasi-Lie algebras, quasi-Hom-Lie algebras, Hom-Lie algebras, Color Lie algebras
4. Examples and constructions of quasi-hom-Lie algebras for discretized derivatives
5. Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$
6. Hom-associative algebras
7. n-ary Hom-Nambu and Hom-Nambu-Lie algebras

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......more........(China)

Motivation

- **Discretizations (quasi-deformations) of vector fields, (quantum) q-deformations of finite-dimensional Lie algebras and infinite-dimensional Lie algebras like Witt and Virasoro algebras, ...** ;
- **q-Deformed vertex operator models of CFT; quantum field theory; quantization**
  1990's – ......
  Lukierski, Kulish, Ellinas, Prischnaider, Isaev,
  Aizawa, Sato, Hu, Liu, Belov, Chaltikian,
  Curtright, Zachos,
  Dobrev, Doebner, Twarock....

More Motivation

- **extensions and q-deformations of differential and homological algebra,**
  differential geometric structures, **non-commutative, twisted differential calculi non-commutative quantum field theory**
  1990's, 2000's – ......
  V. Abramov, O. Liivapuu, R. Kerner, Dimakis, F. Muller-Hoissen, Lychagin, Huru, Jean-Christophe Wallet, M. Dubois-Violette, Axel de Goursaca, Thierry Massona, Kapranov, Kassel, Kac, Borowec, ...

More motivation

- q-**deformed Heisenberg (Weyl) algebras, quantum oscillator algebras, quantum algebras, braided Lie algebras** 1990's ...
  Hellstrom and Silvestrov (book, World Scientific 2000),
  Kulish, Damaskinsky, Fairlie, Curtright, Zachos,
  Michel Rausch de Traubenberg, ...
  Gurevich, Majid, Lychagin, Huru, ... **(braided Lie algebras)**
- q-**analysis,** q-**special functions**
  (1850's –...– 1910's, ... Euler, Gauss, Jackson, ...)

More motivation

- **Color Lie (super)algebras (Γ-graded ϵ-Lie algebras), in particular Lie Super algebras**
  1978 – .... Lukierski, Rittenberg, Wyler, Scheunert, Marcinek, Kwasniewski, Bachturin, Mikhalev, ...
  ... **Sergei Silvestrov (1992, 1994, 1996, 2005: 7 papers, classification, involutoins, representations, ...)**
  2008 – Jean-Christophe Wallet, Axel de Goursaca, Thierry Massona, Michel Rausch de Traubenberg
- **non-associative algebras**
- **non-commutative geometry**
  (algebraic, differential, ...)

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σ-derivations (twisted or deformed derivations)

$\mathcal{A}$ – (commutative) associative $\mathbb{K}$-algebra with unity
$\sigma : \mathcal{A} \to \mathcal{A}$ algebra endomorphism
σ-**derivations**

- $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ linear map
- twisted (deformed) Leibniz rule

$$\partial_\sigma(a \cdot b) = \partial_\sigma(a) \cdot b + \sigma(a) \cdot \partial_\sigma(b)$$

Examples of σ-derivations

- the ordinary derivation operator
  $(\partial\, a)(x) = \lim\limits_{y\to x}\frac{a(y)-a(x)}{y-x} = \frac{da}{dx}(x) = a'(x)$
  $(\partial\, ab)(x) = (\partial a)(x)b(x) + a(x)(\partial b)(x)$
  $\sigma = \mathrm{id} : a(x) \mapsto a(x)$
- Shifted difference operators
  $(\partial\, a)(x) = a(x+h) - a(x)$
  $(\partial\, ab)(x) = (\partial a)(x)b(x) + a(x+h)(\partial b)(x)$
  $\sigma(a)(x) = a(x+h)$
- q-difference operator
  $(\partial\, a)(x) = a(qx) - a(x)$
  $(\partial\, ab)(x) = (\partial\, a)(x)b(x) + a(qx)(\partial\, b)(x)$
  $\sigma(a)(x) = a(qx)$

Examples of σ-derivations

- Jackson q-derivative
  $(\partial\, a)(x) = (D_q a)(x) = \frac{a(qx)-a(x)}{qx-x}$
  $(\partial\, ab)(x) = (\partial a)(x) b(x) + a(qx)(\partial b)(x)$
  $\sigma(a)(x) = a(qx)$
  $\lim_{q\to 1} D_q(a)(x) = a'(x)$
- **"General" σ-derivations (twisted difference operators)**
  $\Omega \subset \mathbb{K}$ any subset of a field
  $T : \Omega \to \Omega$
  Any transformation without fixed points in Ω
  A any algebra of functions a on Ω such that
  $\sigma(a)(x) = a(T(x)) \in A$
  $\partial_\sigma : a(x) \mapsto \frac{a(T(x))-a(x)}{T(x)-x} = \left(\frac{(\sigma - id)}{(T-id)} a\right)(x)$
  $\Downarrow$
  $\partial_\sigma(a \cdot b) = \partial_\sigma(a) \cdot b + \sigma(a) \cdot \partial_\sigma(b)$

σ-derivations on UFD (unique factorization domain)

Theorem 1 $\mathcal{A}$ is UFD

$$\Downarrow$$

Space of all σ-derivations $\mathfrak{D}_\sigma(\mathcal{A})$ is a free rank one $\mathcal{A}$-module with generator

$$\Delta = \frac{(\mathrm{id} - \sigma)}{g} : \quad a \longmapsto \frac{(\mathrm{id} - \sigma)(a)}{g}$$

where $g = \gcd\big((\mathrm{id} - \sigma)(\mathcal{A})\big)$

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Quasi-Hom-Lie algebras of twisted (deformed) vector fields

$\mathcal{A}$ commutative algebra, $\sigma : \mathcal{A} \to \mathcal{A}$ algebra endomorphism, $\Delta : \mathcal{A} \to \mathcal{A}$ σ-derivation,

$Ann(\Delta) = \{a \in A \mid a\Delta = 0\}$, σ-twisted vector fields $\mathcal{A} \cdot \Delta$

Theorem 2 (Hartwig, Larsson, Silvestrov, 2003)

J. of Algebra, 295 (2006), 314-361, Preprint Institute Mittag-Lefler, 2003, Preprint Lund University 2003

Bracket on $\mathcal{A} \cdot \Delta$ (well-defined if $\sigma(Ann(\Delta)) \subseteq Ann(\Delta)$)

$$\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = (\sigma(a) \cdot \Delta)(b \cdot \Delta) - (\sigma(b) \cdot \Delta)(a \cdot \Delta)$$

Closure $\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = \big(\sigma(a)\Delta(b) - \sigma(b)\Delta(a)\big) \cdot \Delta$

Skew-symmetry $\langle a \cdot \Delta, b \cdot \Delta \rangle_\sigma = -\langle b \cdot \Delta, a \cdot \Delta \rangle_\sigma$

Twisted 6 term Jacobi Identity $\Delta \circ \sigma(a) = \delta \cdot \sigma \circ \Delta(a)$, $\delta \in A$

$$\circlearrowleft_{a,b,c} \Big(\big\langle \sigma(a) \cdot \Delta, \langle b \cdot \Delta, c \cdot \Delta \rangle_\sigma \big\rangle_\sigma + \delta \cdot \big\langle a \cdot \Delta, \langle b \cdot \Delta, c \cdot \Delta \rangle_\sigma \big\rangle_\sigma \Big) = 0$$

$\mathcal{A}$ is UFD $\Rightarrow$ $\delta = \frac{\sigma(g)}{g}$, $g = GCD(id - \sigma)(\mathcal{A})$

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Quasi-Lie algebra

D. Larsson, S. Silvestrov, Quasi-Lie algebras, Contemporary Mathematics, Vol. 391, 2005.

$(L, \langle \cdot, \cdot \rangle_L, \alpha, \beta, \omega, \theta)$

1. L is a linear space over $\mathbb{F}$,
2. $\langle \cdot, \cdot \rangle_L : L \times L \to L$ is a bilinear product or bracket in L;
3. $\alpha, \beta : L \to L$, are linear maps,
4. $\omega : D_\omega \to \mathcal{L}_{\mathbb{F}}(L)$ and $\theta : D_\theta \to \mathcal{L}_{\mathbb{F}}(L)$ are maps with domains of definition $D_\omega, D_\theta \subseteq L \times L$,

ω-Symmetry

$\langle x, y \rangle_L = \omega(x, y) \langle y, x \rangle_L$, for all $(x, y) \in D_\omega$

Quasi-Jacobi identity

$\circlearrowleft_{x,y,z} \left\{ \theta(z, x) \big(\langle \alpha(x), \langle y, z \rangle_L \rangle_L + \beta \langle x, \langle y, z \rangle_L \rangle_L \big) \right\} = 0$

for all $(z, x), (x, y), (y, z) \in D_\theta$

Graded (color) Quasi-Lie algebra

D. Larsson, S. Silvestrov, Graded quasi-Lie algebras, Czechoslovak Journal of Physics, 55, 11 (2005), 1473-1478.

Γ-graded (color) quasi-Lie algebra

$(\Gamma, \hat{+})$ commutative semigroup

$(L, \langle\cdot,\cdot\rangle_L, \alpha, \beta, \omega, \theta)$

1. $L = \bigoplus_{\gamma\in\Gamma} L_\gamma$ is a Γ-graded linear space over $\mathbb{F}$,
2. $\langle\cdot,\cdot\rangle_L : L \times L \to L$ is a bilinear map (bracket);
3. $\alpha, \beta : L \to L$ are linear maps mapping $\cup_{\gamma\in\Gamma} L_\gamma$ to $\cup_{\gamma\in\Gamma} L_\gamma$.
   $\omega : D_\omega \to \mathcal{L}_{\mathbb{F}}(L)$, $\theta : D_\theta \to \mathcal{L}_{\mathbb{F}}(L)$,
   $D_\omega, D_\theta \subseteq \cup_{\gamma\in\Gamma} L_\gamma \times \cup_{\gamma\in\Gamma} L_\gamma$,

Γ-**grading axiom** $\langle L_{\gamma_1}, L_{\gamma_2}\rangle_L \subseteq L_{\gamma_1 \hat{+} \gamma_2}$ for all $\gamma_1, \gamma_2 \in \Gamma$;

ω-**symmetry** $\langle x, y\rangle_L = \omega(x,y)\langle y, x\rangle_L$, for all $(x,y) \in D_\omega$;

quasi-Jacobi identity

$\circlearrowleft_{x,y,z} \left\{ \theta(z,x)\big(\langle \alpha(x), \langle y, z\rangle_L\rangle_L + \beta\langle x, \langle y, z\rangle_L\rangle_L\big)\right\} = 0$, for $(z,x) \in D_\theta$, $(x,y) \in D_\theta$, $(y,z) \in D_\theta$.

Note: $(\omega(x,y)\omega(y,x) - \mathrm{id})\langle x, y\rangle = 0$, if $(x,y), (y,x) \in D_\omega$ by ω-symmetry

Hom-Lie color algebra

Hom-Lie color algebras is subclass of graded (color) quasi Lie algebras.
Hom-Lie superalgebras $\Gamma = \mathbb{Z}_2$ and $\varepsilon(a,b) = (-1)^{|a||b|}$
$(L, [\cdot,\cdot], \alpha, \varepsilon)$
L is Γ-graded space
$[\cdot,\cdot] : L \times L \to L$ is an even bilinear mapping
α is an even linear map
ε is bi-character on Γ
ε-skew-symmetry $[x,y] = -\varepsilon(x,y)[y,x]$,
Hom ε-Jacobi identity
$\circlearrowleft_{x,y,z} \varepsilon(z,x)[\alpha(x),[y,z]\,] = 0$,
for all homogenous elements x, y, z in L.

Quasi-Hom-Lie algebras

$(L, \langle \cdot, \cdot \rangle_L, \alpha, \beta, \omega)$

1. L is a linear space over field $\mathbb{K}$
2. $\langle \cdot, \cdot \rangle_L : L \times L \to L$ is a bilinear map
3. $\alpha, \beta : L \to L$ are linear maps
4. $\omega : D_\omega \to End(L)$ is a map with domain of definition $D_\omega \subseteq L \times L$

Quasi-Hom-Lie algebras

- (**β-twisting**) The map α is a β-twisted algebra homomorphism,

$$\langle \alpha(x), \alpha(y) \rangle_L = \beta \circ \alpha \langle x, y \rangle_L,$$

  for all $x, y \in L$
- (**ω-symmetry**) $\langle x, y \rangle_L = \omega(x, y) \langle y, x \rangle_L$, for all $(x, y) \in D_\omega$
- **Quasi-Hom-Lie Jacobi identity**

$$\circlearrowleft_{x,y,z} \left\{ \omega(z,x) \Big(\langle \alpha(x), \langle y, z \rangle_L \rangle_L + \beta \langle x, \langle y, z \rangle_L \rangle_L \Big) \right\} = 0$$

  for all $(z, x), (x, y), (y, z) \in D_\omega$

Hom-Lie algebras

$\beta = \mathrm{id}_L, \qquad \omega = -\mathrm{id}_L$

1. a linear map $\alpha : L \to L$
2. bilinear multiplication (bracket) $\langle \cdot, \cdot \rangle_\alpha$ such that

- **skew-symmetry**

$$\langle x, y \rangle_\alpha = -\langle y, x \rangle_\alpha$$

- **Hom-Lie Jacobi identity**

$$\circlearrowleft_{x,y,z} \left\langle \alpha(x), \langle y, z \rangle_\alpha \right\rangle_\alpha = 0$$

for all $x, y, z \in L$

Γ-graded ε-Lie algebras (Color Lie algebras)

Γ – **commutative group (or semigroup).**
K field of $char K \neq 2, 3$
Γ-**graded algebra**

$$L = \bigoplus_{\gamma \in \Gamma} L_\gamma$$

$\langle \cdot, \cdot \rangle : L \times L \longrightarrow L$
$\forall \quad A \in L_\alpha, B \in L_\beta, C \in L_\gamma, \quad \alpha, \beta, \gamma \in \Gamma$:

$$\langle A, B \rangle = -\varepsilon(\alpha, \beta) \langle B, A \rangle$$ (ε-skew symmetry)

$$\varepsilon(\gamma, \alpha)\langle A, \langle B, C \rangle\rangle + \varepsilon(\beta, \gamma)\langle C, \langle A, B \rangle\rangle + \varepsilon(\alpha, \beta)\langle B, \langle C, A \rangle\rangle = 0$$

(ε-Jacoby identity)

Color Lie algebras are examples of quasi Hom-Lie algebras.

L Γ-graded quasi Hom-Lie algebra
$L = \oplus_{\gamma\in\Gamma} L_\gamma$
$\alpha = \beta = \mathrm{id}_L$, $\omega(x,y)v = -\varepsilon(\gamma_x,\gamma_y)v$
$v \in L$ and $(x,y) \in D_\omega = \cup_{\gamma\in\Gamma} L_\gamma$
$\gamma_x, \gamma_y \in \Gamma$ graded degrees of x and y.
The ω-symmetry and the qhl-Jacobi identity

$$\downarrow$$

Γ-Graded ε-symmetry and ε-Jacobi identities for color Lie algebras.
Lie superalgebras
$\Gamma = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$,
$\varepsilon(\gamma_x,\gamma_y) = (-1)^{\gamma_x\gamma_y}$
$\gamma_x\gamma_y$ is the product in $\mathbb{Z}_2$.

Quasi-Leibniz-Loday algebra

D. Larsson, S. Silvestrov, Quasi-Lie algebras, Contemporary Mathematics, Vol. 391, 2005.

Leibniz-Loday algebra

$$\langle\langle x, y\rangle, z\rangle = \langle\langle x, z\rangle, y\rangle + \langle x, \langle y, z\rangle\rangle. \quad (1)$$

Quasi-Leibniz-Loday algebra

$$\begin{aligned}
\theta(y,z)\big(\omega(\alpha(z), \langle x,y\rangle)\langle\langle x,y\rangle, \alpha(z)\rangle + \beta\circ\omega(z,\langle x,y\rangle)\langle\langle x,y\rangle, z\rangle\big) = \\
= -\theta(x,y)\big(\omega(\alpha(y), \langle z,x\rangle)\langle\omega(z,x)\langle x,z\rangle, \alpha(y)\rangle + \\
+ \beta\circ\omega(y,\langle z,x\rangle)\langle\omega(z,x)\langle x,z\rangle, y\rangle\rangle\big) - \\
- \theta(z,x)\big(\langle\alpha(x), \langle y,z\rangle\rangle + \beta\langle x, \langle y,z\rangle\rangle\big)
\end{aligned}$$

$(z,x), (x,y), (y,z) \in D_\theta$,
$(\alpha(z), \langle x,y\rangle), (\alpha(y), \langle z,x\rangle), (y, \langle z,x\rangle), (z,x) \in D_\omega$
$\alpha = \beta = \mathrm{id}$ and $\theta = \omega = -\mathrm{id}$, one recovers (1)

Hom-Leibniz algebras (Hom-Loday algebras) (special case of quasi-Leibniz algebras)

Definition $(V, [\cdot,\cdot], \alpha)$ consisting of a linear space V, bilinear map $[\cdot,\cdot] : V \times V \to V$ and a homomorphism $\alpha : V \to V$ satisfying

$$[[x, y], \alpha(z)] = [[x, z], \alpha(y)] + [\alpha(x), [y, z]].$$

If a Hom-Leibniz algebra is skewsymmetric then it is a Hom-Lie algebra.

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q-deformed Witt algebra Witt_q

$$\mathfrak{D}_\sigma(\mathcal{A}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot d_n$$

$\Delta = tD_q = \frac{\sigma - \mathrm{id}}{q-1} : f(t) \mapsto \frac{f(qt)-f(t)}{q-1}$
$\sigma(t) = qt, \quad \sigma(f)(t) = f(qt), \quad \{n\}_q = \frac{q^n-1}{q-1}$
Skew-symmetric product. $d_n = -t^n\Delta$

$$\langle d_n, d_m\rangle = q^n d_n d_m - q^m d_m d_n = \big(\{n\}_q - \{m\}_q\big) d_{n+m}$$

Graded Hom-Lie algebra $\langle L_n, L_m\rangle \subseteq L_{n+m}$
Hom-Lie algebra Jacobi-identity
$\circlearrowleft_{n,m,l} (q^n+1)\big\langle d_n, \langle d_m, d_l\rangle\big\rangle = 0$
$\alpha(d_n) = (q^n+1)d_n$
$\circlearrowleft_{n,m,l} \big\langle \alpha(d_n), \langle d_m, d_l\rangle\big\rangle = 0$

q-deformed Virasoro algebra. Hom-Lie central extension

$$(\mathrm{Vir}_q, \hat{\sigma}) = (\mathrm{Witt}_q \oplus \mathbb{C} \cdot \mathbf{c}, \hat{\sigma}) \quad \{d_n : n \in \mathbb{Z}\} \cup \{\mathbf{c}\}$$

$$\hat{\sigma} : \mathrm{Vir}_q \to \mathrm{Vir}_q, \quad \hat{\sigma}(d_n) = q^n d_n, \quad \hat{\sigma}(\mathbf{c}) = \mathbf{c}$$

$$\langle d_n, d_m \rangle = \big(\{n\}_q - \{m\}_q\big) d_{n+m} + \\ + \delta_{n+m,0} \frac{q^{-n}}{6(1+q^n)} \{n-1\}_q \{n\}_q \{n+1\}_q \mathbf{c}$$

$$\langle \mathbf{c}, \mathrm{Vir}_q \rangle = 0$$

Loop quasi Hom-Lie algebras

Quasi-Hom-Lie algebra $\mathfrak{g}$

$$\Downarrow$$

Linear space

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{K}[t, t^{-1}]$$

The algebra of Laurent polynomials with coefficients in the qhl-algebra $\mathfrak{g}$.

$\alpha_{\hat{\mathfrak{g}}} := \alpha_{\mathfrak{g}} \otimes \mathrm{id}$
$\beta_{\hat{\mathfrak{g}}} := \beta_{\mathfrak{g}} \otimes \mathrm{id}$
$\omega_{\hat{\mathfrak{g}}} := \omega_{\mathfrak{g}} \otimes \mathrm{id}$
$\langle x \otimes t^n, y \otimes t^m \rangle_{\hat{\mathfrak{g}}} = \langle x, y \rangle_{\mathfrak{g}} \otimes t^{n+m}$
$\hat{\mathfrak{g}}$ is a quasi Hom-Lie algebra.

Non-linear Quasi-Lie deformations of Witt algebra

$$\mathfrak{D}_\sigma(\mathcal{A}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot d_n,$$

$$D = \alpha t^{-k+1}\frac{\mathrm{id} - \sigma}{t - qt^s}, \quad \sigma(t) = qt^s$$

Skew-symmetric product $\qquad d_n = -t^n D$

$$\langle d_n, d_m\rangle_\sigma = q^n d_{ns} d_m - q^m d_{ms} d_n$$

$$\langle d_n, d_m\rangle_\sigma = \text{linear combinations of generators}$$

For $n, m \geq 0$:

$$\langle d_n, d_m\rangle_\sigma = \alpha \mathrm{sign}(n-m)\sum_{l=\min(n,m)}^{\max(n,m)-1} q^{n+m-1-l} d_{s(n+m-1)-(k-1)-l(s-1)}$$

Non-linear Quasi-Lie deformations of Witt algebra

σ-deformed Jacobi-identity

$$\circlearrowleft_{n,m,l}\ \left(\left\langle q^n d_{ns}, \langle d_m, d_l\rangle_\sigma\right\rangle_\sigma + \underbrace{q^k t^{k(s-1)} \sum_{r=0}^{s-1} (qt^{s-1})^r}_{=\delta} \left\langle d_n \langle d_m, d_l\rangle_\sigma\right\rangle_\sigma\right) = 0.$$

Quasi-Hom-Lie algebra, not Hom-Lie algebra for $s \neq 1$

Other non-linear Quasi-Lie deformations of Witt algebra

$\sigma(t) = qt^s$

$$D = \frac{\mathrm{id} - \sigma}{\eta^{-1} \cdot t^k}$$

generates a cyclic $\mathcal{A}$-submodule $\mathfrak{M}$ of $\mathfrak{D}_\sigma(\mathcal{A})$, proper for $s \neq 1$
($s \neq 1$: $\sigma(t) = \beta t$ for some $\beta \in \mathbb{K}$)
Theorem The linear space

$$\mathfrak{M} = \bigoplus_{i \in \mathbb{Z}} \mathbb{K} \cdot d_i \quad \text{with} \quad d_i = -t^i D$$

is a quasi-Lie algebra

$$\langle d_n, d_m \rangle_\sigma = q^n d_{ns} d_m - q^m d_{ms} d_n = \eta q^m d_{ms+n-k} - \eta q^n d_{ns+m-k}$$

$s \in \mathbb{Z}$ and $\eta \in \mathbb{C}$

Other non-linear Quasi-Lie deformations of Witt algebra

The σ-deformed Jacobi identity

$$\circlearrowleft_{n,m,l} \left(\left\langle q^n d_{ns}, \langle d_m, d_l \rangle_\sigma \right\rangle_\sigma + \underbrace{q^k t^{(s-1)k}}_{=\delta} \left\langle d_n, \langle d_m, d_l \rangle_\sigma \right\rangle_\sigma \right) = 0$$

$q = 1$, $k = 0$ **and** $s = 1$
get a commutative algebra with countable number of generators instead of the Witt algebra.

Non-linear Quasi-Lie deformations of Witt algebra are almost graded

Almost graded algebras $s = 1$:
$\langle L_n, L_m \rangle_\sigma \subseteq L_{n+m-k}$
(quasi-Lie deformations of) Krichever-Novikov type algebras,
Graded $k = 0, s = 1$: $\langle L_n, L_m \rangle_\sigma \subseteq L_{n+m}$
Hyper almost Graded algebras:

$$\langle L_n, L_m \rangle_\sigma \subseteq \bigoplus_{j \in \mathbb{Z} \cap [ms+n-k, ns+m-k]} L_j$$

$ms + n - k = m + n + m(s-1) - k$

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Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$

Daniel Larsson, Sergei Silvestrov, (2005) Communications in Algebra, 35, 4303-4318, 2007.

D. Larsson, S. Silvestrov, The Lie algebra $sl_2(F)$ and quasi-deformations, Czechoslovak Journal of Physics, **55**, 11 (2005), 1467-1472.

D. Larsson, G. Sigurdsson, S. Silvestrov, On some almost quadratic algebras coming from twisted derivations, Preprints in mathematical sciences (2006:9), LUTFMA-5073-2006, Centre for Mathematical Sciences, Lund University. 11 pp. Proceedings of AGMF network workshop, Tallinn 2005, J. Nonlin. Math. Phys. Vol. 13, 2006.

D. Larsson, S. D. Silvestrov, Quasi-deformations of $sl_2(F)$ with base $\mathbb{R}$[t,t-1]. Czechoslovak J. Phys. 56 (2006), no. 10-11, 1227–1230

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$

$$\mathfrak{sl}_2(\mathbb{K}) : [\mathbf{h}, \mathbf{e}] = 2\mathbf{e},\ [\mathbf{h}, \mathbf{f}] = -2\mathbf{f},\ [\mathbf{e}, \mathbf{f}] = \mathbf{h}$$

Representation

$\mathbf{e} \mapsto \partial$, $\mathbf{h} \mapsto -2t\partial$, $\mathbf{f} \mapsto -t^2\partial$

Lie algebra product $[a, b] = ab - ba$

σ-Twisted vector fields

$\mathbf{e} \mapsto \partial_\sigma$, $\mathbf{h} \mapsto -2t\partial_\sigma$, $\mathbf{f} \mapsto -t^2\partial_\sigma$

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$

$$\begin{aligned}\langle a\cdot\Delta, b\cdot\Delta\rangle_\sigma &= (\sigma(a)\cdot\Delta)(b\cdot\Delta)-(\sigma(b)\cdot\Delta)(a\cdot\Delta)\\ &= \big(\sigma(a)\Delta(b)-\sigma(b)\Delta(a)\big)\cdot\Delta\end{aligned}$$

Assumption $\sigma(1)=1$ and $\partial_\sigma(1)=0$

$$\Downarrow$$

$$\begin{aligned}\langle \mathbf{h},\mathbf{f}\rangle &= 2\sigma(t)\partial_\sigma(t)t\partial_\sigma\\ \langle \mathbf{h},\mathbf{e}\rangle &= 2\partial_\sigma(t)\partial_\sigma\\ \langle \mathbf{e},\mathbf{f}\rangle &= -(\sigma(t)+t)\partial_\sigma(t)\partial_\sigma\end{aligned}$$

Closure of the bracket on $L=\mathbb{K}\mathbf{e}\bigoplus\mathbb{K}\mathbf{f}\bigoplus\mathbb{K}\mathbf{h}$

$$\Updownarrow$$

$$\deg\sigma(t)\partial_\sigma(t)t\le 2$$

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$. Affine $\sigma(t)$ and ∂_σ is $c\frac{d}{dx}$-like on t^k

Case 1: $\mathcal{A} = \mathbb{K}[t]$, $\sigma(t) = q_0 + q_1 t$, $\partial_\sigma(t) = p_0$

$$\langle \mathbf{h}, \mathbf{f} \rangle : \ -2q_0\mathbf{ef} + q_1\mathbf{hf} + q_0^2\mathbf{eh} - q_0q_1\mathbf{h^2} - q_1^2\mathbf{fh} = -q_0p_0\mathbf{h} - 2q_1p_0\mathbf{f}$$

$$\langle \mathbf{h}, \mathbf{e} \rangle : \ -2q_0\mathbf{e^2} + q_1\mathbf{he} - \mathbf{eh} = 2p_0\mathbf{e}$$

$$\langle \mathbf{e}, \mathbf{f} \rangle : \ \mathbf{ef} + q_0^2\mathbf{e^2} - q_0q_1\mathbf{he} - q_1^2\mathbf{fe} = -q_0p_0\mathbf{e} + \frac{q_1+1}{2}p_0\mathbf{h}.$$

$q_1 = 1$, $q_0 = p_0 = 0$ gives $\mathfrak{sl}_2(\mathbb{K})$

${}_q\mathfrak{sl}_2(\mathbb{K})$ Jackson $\mathfrak{sl}_2(\mathbb{K})$ (quasi-Lie algebra). Linear $\sigma(t)$ and ∂_σ is $c\frac{d}{dx}$-like on t^k

$q_0 = 0$, $q = q_1 \neq 0$ $\left(\frac{d}{dt} \mapsto D_q\right)$

$$\begin{aligned} \mathbf{hf} - q\mathbf{fh} &= -2p_0\mathbf{f} \\ \mathbf{he} - q^{-1}\mathbf{eh} &= 2q^{-1}p_0\mathbf{e} \\ \mathbf{ef} - q^2\mathbf{fe} &= \frac{q+1}{2}p_0\mathbf{h} \end{aligned}$$

Iterated Ore extension of $\mathbb{K}[z]$, Auslander-regular, global dimension at most three, has PBW-basis, noetherian domain of GK-dimension three, Koszul as an almost quadratic algebra

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$

$p_0 = 0 \rightarrow$ "abelianized" version. But $\partial_\sigma = 0$

$p_0 = 1$:

$\langle \mathbf{h}, \mathbf{f} \rangle = -2q\mathbf{f}$, $\langle \mathbf{h}, \mathbf{e} \rangle = 2\mathbf{e}$, $\langle \mathbf{e}, \mathbf{f} \rangle = \frac{q+1}{2}\mathbf{h}$

Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$. Twisted (Hom-Lie) Jacobi identity

$$\alpha(\mathbf{e}) = \tfrac{q^{-1}+1}{2}\,\mathbf{e}, \quad \alpha(\mathbf{h}) = \mathbf{h}, \quad \alpha(\mathbf{f}) = \tfrac{q+1}{2}\,\mathbf{f}$$

$$\langle \alpha(\mathbf{e}), \langle \mathbf{f}, \mathbf{h} \rangle\rangle + \langle \alpha(\mathbf{f}), \langle \mathbf{h}, \mathbf{e} \rangle\rangle + \langle \alpha(\mathbf{h}), \langle \mathbf{e}, \mathbf{f} \rangle\rangle = 0$$

Quasi-Lie deformations of $\mathfrak{sl}_2(\mathbb{K})$ on $\mathbb{K}[t]/(t^3)$

$\mathcal{A} = \mathbb{K}[t]/(t^3)$, $\sigma(t) = q_1 t + q_2 t^2$, $\partial_\sigma(t) = p_1 t$.

$$\begin{aligned}
\langle h, f\rangle : &\quad q_1 hf + 2q_2 f^2 - q_1^2 fh = 0\\
\langle h, e\rangle : &\quad q_1 he + 2q_2 fe - eh = -p_1 h - 2p_2 f\\
\langle e, f\rangle : &\quad ef - q_1^2 fe = p_1(q_1+1)f.
\end{aligned}$$

$\mathfrak{sl}_2(\mathbb{K})$ cannot be recovered in any "limit"
$p_1 = 0$ representation collapse $\partial_\sigma(t) = 0$

Quasi-Lie quasi-deformations of $\mathfrak{sl}_2(\mathbb{K})$ on $\mathbb{K}[t]/(t^3)$. Special limits

"non-commutative deformation of $\mathbb{K}[x,y]$ in f-direction"

$q_1 = 1, q_2 = -\frac{1}{2}, p_1 = p_2 = 0$

$\mathbf{hf} - \mathbf{fh} = \mathbf{f}^2, \quad \mathbf{he} - \mathbf{eh} = \mathbf{fe}, \quad \mathbf{ef} - \mathbf{fe} = 0$

$\mathbf{f} = 0 \to \mathbb{K}[h,e]$

Quasi-Lie quasi-deformations of $\mathfrak{sl}_2(\mathbb{K})$ on $\mathbb{K}[t]/(t^3)$. Special limits.

solvable 3-dimensional Lie algebra

$q_1 = 1,\ q_2 = 0,\ p_1 = 1,\ p_2 = a/2$

$\mathbf{hf} - \mathbf{fh} = 0, \mathbf{he} - \mathbf{eh} = -\mathbf{h} - a\mathbf{f}, \mathbf{ef} - \mathbf{fe} = 2\mathbf{f}$

Heisenberg Lie algebra

$p_1 = 0, p_2 = -1/2$

$\mathbf{hf} - \mathbf{fh} = 0, \mathbf{he} - \mathbf{eh} = \mathbf{f}, \mathbf{ef} - \mathbf{fe} = 0$

Polynomials in 3 commuting variables

$q_1 = 1,\ q_2 = 0,\ p_1 = p_2 = 0 \to \mathbb{K}[x, y, z]$

Quasi-Hom-Lie algebra Jacobi identity. Case $q_1 p_1 \neq 0$

$$\circlearrowleft_{x,y,z} \left(\langle \sigma(x), \langle y, z \rangle \rangle + \underbrace{(1 - \frac{q_1 p_2 - p_2 - p_1 q_2}{p_1} t + \xi_2 t^2)}_{=\delta} \langle x, \langle y, z \rangle \rangle \right) = 0$$

Quasi-Lie deformations on the algebra $\mathbb{K}[t]/(t^N)$

$\mathbb{K}$ include all N^{th}-roots of unity
$\mathcal{A} = \mathbb{K}[t]/(t^N)$ for $N \geq 2$ N-dimensional $\mathbb{K}$-vector space and a finitely generated $\mathbb{K}[t]$-module with basis $\{1, t, \ldots, t^{N-1}\}$.

Quasi-Lie deformations on the algebra $\mathbb{K}[t]/(t^N)$

$$\partial_\sigma(t) = p(t) = \sum_{k=0}^{N-1} p_k t^k, \quad \sigma(t) = \sum_{k=0}^{N-1} q_k t^k \tag{2}$$

considering these as elements in the ring $\mathbb{K}[t]/(t^N)$.
$t^N = 0$ in $\mathbb{K}[t]/(t^N)$

Quasi-Lie deformations on the algebra $\mathbb{K}[t]/(t^N)$

Commutation relations

$g_i = c_i t^i \partial_\sigma,\ c_i \in \mathbb{K},\ c_i \neq 0.$

The bracket is closed on linear span of g_i's For $N-1 \geq i, j \geq 0$

$$\langle g_i, g_j \rangle = c_i c_j [\partial_\sigma(t^j)\sigma(t)^i - \sigma(t)^j \partial_\sigma(t^i)]\partial_\sigma$$

$$= c_i c_j \sum_{k=0}^{|j-i|-1} \mathrm{sign}(j-i) \sum_{\substack{k_1, k_2, \ldots, k_{N-1} \geq 0 \\ k_1+k_2+\ldots+k_{N-1} = k+\min\{i,j\} \\ k_2+2k_3+\ldots+(N-2)k_{N-1} < N}} \frac{(k+\min\{i,j\})!}{k_1! k_2! \cdots k_{N-1}!}$$

$$\times\, q_1^{k_1} q_2^{k_2} \ldots q_{N-1}^{k_{N-1}} t^{k_2+2k_3+\ldots+(N-2)k_{N-1}} \sum_{l=0}^{N-1} p_l t^{i+j+l-1} \partial_\sigma$$

Quasi-Lie deformations on the algebra $\mathbb{K}[t]/(t^N)$

$$= c_i c_j \sum_{l=0}^{N-1} p_l \sum_{k=0}^{|j-i|-1} \operatorname{sign}(j-i) \sum_{\substack{k_1,k_2,\ldots,k_{N-1}\geq 0 \\ k_1+k_2+\ldots+k_{N-1}=k+\min\{i,j\} \\ k_2+2k_3+\ldots+(N-2)k_{N-1}\leq N-i-j-l}} \frac{(k+\min\{i,j\})!}{k_1!k_2!\cdots k_{N-1}!}$$

$$\times\, q_1^{k_1} q_2^{k_2} \ldots q_{N-1}^{k_{N-1}} \frac{g_{i+j+l-1+k_2+2k_3+\ldots+(N-2)k_{N-1}}}{c_{i+j+l-1+k_2+2k_3+\ldots+(N-2)k_{N-1}}}$$

where $\operatorname{sign}(x) = -1$ if $x < 0$, $\operatorname{sign}(x) = 0$ if $x = 0$ and $\operatorname{sign}(x) = 1$ if $x > 0$.

Outline

1. σ-derivations (twisted or deformed or discretized derivations)
2. Quasi-Hom-Lie algebras of twisted (deformed) vector fields
3. Quasi-Lie algebras, quasi-Hom-Lie algebras, Hom-Lie algebras, Color Lie algebras
4. Examples and constructions of quasi-hom-Lie algebras for discretized derivatives
5. Quasi-Lie (quasi-)deformations of $\mathfrak{sl}_2(\mathbb{K})$
6. **Hom-associative algebras**
7. n-ary Hom-Nambu and Hom-Nambu-Lie algebras

Hom-associative algebras ↦ Hom-Lie algebras

Hom-associative algebra (V, μ, α)
V linear space, $\mu : V \times V \to V$ bilinear map, $\alpha : V \to V$ linear map

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))$$

$$\alpha(x)yz = xy\alpha(z)$$

Hom-associative algebras $\mapsto$ Hom-Lie algebras

Theorem

To any Hom-associative algebra (V, μ, α), one may associate a Hom-Lie algebra defined for all $x, y \in V$ by the bracket $[x, y] = \mu(x, y) - \mu(y, x)$.

Hom-associative algebras are **Hom-Lie admissible**

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

G subgroup of the permutations group $\mathcal{S}_3$
Definition Hom-algebra (V, μ, α) is said to be G-**Hom-associative** if

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} (\mu(\mu(x_{\sigma(1)}, x_{\sigma(2)}), \alpha(x_{\sigma(3)})) - \mu(\alpha(x_{\sigma(1)}), \mu(x_{\sigma(2)}, x_{\sigma(3)})) = 0$$

$x_i \in V$
$(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ.

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

Equivalently

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} a_{\mu,\alpha} \circ \sigma = 0$$

$\sigma(x_1, x_2, x_3) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.

Theorem

Any G-Hom-associative algebra
is a Hom-Lie admissible algebra.

G-Hom-associative algebras

The subgroups of $\mathcal{S}_3$ are
$G_1 = \{Id\}, \quad G_2 = \{Id, \tau_{12}\}, \quad G_3 = \{Id, \tau_{23}\}$
$G_4 = \{Id, \tau_{13}\}, \quad G_5 = A_3, \quad G_6 = \mathcal{S}_3$
A_3 is the alternating group;
τ_{ij} is the transposition of i and j.

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

- The G_1-Hom-associative algebras are the Hom-associative algebras.
- The G_2-Hom-associative algebras satisfy

$$\mu(\alpha(x),\mu(y,z))-\mu(\alpha(y),\mu(x,z))=\mu(\mu(x,y),\alpha(z))-\mu(\mu(y,x),\alpha(z))$$

  When α is the identity the algebra is called Vinberg algebra or left symmetric algebra.
- The G_3-Hom-associative algebras satisfy

$$\mu(\alpha(x),\mu(y,z))-\mu(\alpha(x),\mu(z,y))=\mu(\mu(x,y),\alpha(z))-\mu(\mu(x,z),\alpha(y))$$

  When α is the identity the algebra is called pre-Lie algebra or right symmetric algebra.

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

- The G_4-Hom-associative algebras satisfy

$$\mu(\alpha(x),\mu(y,z))-\mu(\alpha(z),\mu(y,x))=\mu(\mu(x,y),\alpha(z))-\mu(\mu(z,y),\alpha(x))$$

- The G_5-Hom-associative algebras satisfy the condition

$$\mu(\alpha(x),\mu(y,z))+\mu(\alpha(y),\mu(z,x))+\mu(\alpha(z),\mu(x,y))= \\ \mu(\mu(x,y),\alpha(z))+\mu(\mu(y,z),\alpha(x))+\mu(\mu(z,x),\alpha(y))$$

  If the product μ is skewsymmetric, then this condition is the Hom-Jacobi identity.
- The G_6-Hom-associative algebras are the Hom-Lie admissible algebras.

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

A Hom-pre-Lie algebra is a triple (V, μ, α) consisting of a linear space V, a bilinear map $\mu : V \times V \to V$ and a homomorphism α satisfying

$$\mu(\alpha(x), \mu(y, z)) - \mu(\alpha(x), \mu(z, y)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(x, z), \alpha(y))$$

G-Hom-associative algebras $\mapsto$ Hom-Lie algebras

Theorem

Any G-Hom-associative algebra is a Hom-Lie admissible algebra.

From Lie algebras to Hom-Lie algebras. Composition trick

$(V, [\cdot,\cdot])$ Lie algebra
$\alpha : V \to V$ Lie algebra endomorphism
Then $(V, [\cdot,\cdot]_\alpha)$ is a Hom-Lie algebra

$$[x,y]_\alpha = \alpha([x,y])$$

$$[x,y]_\alpha = -[y,x]_\alpha, \qquad \circlearrowleft_{x,y,z} [[\alpha(x),[y,z]_\alpha]_\alpha = 0.$$

Hom-Leibniz algebras (Hom-Loday algebras) (special case of quasi-Leibniz algebras)

Definition $(V, [\cdot,\cdot], \alpha)$ consisting of a linear space V, bilinear map $[\cdot,\cdot] : V \times V \to V$ and a homomorphism $\alpha : V \to V$ satisfying

$$[[x, y], \alpha(z)] = [[x, z], \alpha(y)] + [\alpha(x), [y, z]].$$

If a Hom-Leibniz algebra is skewsymmetric then it is a Hom-Lie algebra.

Hom-Poisson algebra

Definition

$(V, \mu, \{\cdot, \cdot\}, \alpha)$

V linear space, $\mu : V \times V \to V$ and $\{\cdot, \cdot\} : V \times V \to V$ bilinear maps

$\alpha : V \to V$ linear map:

1) (V, μ, α) is a commutative Hom-associative algebra

2) $(V, \{\cdot, \cdot\}, \alpha)$ is a Hom-Lie algebra

3) for all x, y, z in V,

$$\{\alpha(x), \mu(y, z)\} = \mu(\alpha(y), \{x, z\}) + \mu(\alpha(z), \{x, y\}).$$

Hom-Poisson algebra

Equivalently:

$$\{\mu(x,y),\alpha(z)\} = \mu(\{x,z\},\alpha(y)) + \mu(\alpha(x),\{y,z\})$$

for all x, y, z in V.
$ad_z(\cdot) = \{\cdot, z\}$ is a Hom-derivation for the multiplication μ

Hom-Poisson algebra

Let $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ be a deformation of the commutative Hom-associative algebra
$\mathcal{A}_0 = (V, \mu_0, \alpha_0)$

$$\mu_t(x,y) = \mu_0(x,y) + \mu_1(x,y)t + \mu_2(x,y)t^2 + \dots$$

Then

$$\frac{\mu_t(x,y) - \mu_t(y,x)}{t} = \mu_1(x,y) - \mu_1(y,x) + t\sum_{i\geq 2}(\mu_i(x,y) - \mu_i(y,x))t^{i-2}$$

Hence, if t goes to zero then $\frac{\mu_t(x,y)-\mu_t(y,x)}{t}$ goes to
$\{x,y\} := \mu_1(x,y) - \mu_1(y,x)$

Hom-Poisson algebra

Theorem
$\mathcal{A}_0 = (V, \mu_0, \alpha_0)$
a commutative Hom-associative algebra
$\mathcal{A}_t = (V, \mu_t, \alpha_t)$ a deformation of $\mathcal{A}_0$.
Consider the bracket
$\{x, y\} = \mu_1(x, y) - \mu_1(y, x)$
is the first order element of the deformation μ_t.

$$\Downarrow$$

$(V, \mu_0, \{,\}, \alpha_0)$ is a Hom-Poisson algebra.

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n-ary Hom-Nambu and Hom-Nambu-Lie algebras

H. Ataguema, A. Makhlouf, S. Silvestrov, Generalization of n-ary Nambu Algebras and Beyond, Journal of Mathematical Physics, 50, 083501,2009

Definition

An *n-ary Hom-Nambu algebra* is a triple $(V, [\cdot, \dots, \cdot], \alpha)$, consisting of a vector space V, an n-linear map $[\cdot, \cdots, \cdot] : V^{\times n} \rightarrow V$ and a family $\alpha = (\alpha_i)_{i=1,\cdots,n-1}$ of linear maps $\alpha_i : V \rightarrow V$, $i = 1, \cdots, n-1$ satisfying

The n-ary Hom-Nambu identity

$$[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [x_n, \dots, x_{2n-1}]] =$$

$$\sum_{i=n}^{2n-1} [\alpha_1(x_n), \dots, \alpha_{i-n}(x_{i-1}), [x_1, \cdots, x_{n-1}, x_i], \alpha_{i-n+1}(x_{i+1}) \dots, \alpha_{n-1}(x_{2n-1})]$$

for all $(x_1, \cdots, x_{2n-1}) \in V^{2n-1}$.

n-ary Hom-Nambu and Hom-Nambu-Lie algebras

Ternary Hom-Nambu algebras

$$\begin{aligned}[\alpha_1(x_1), \alpha_2(x_2), [x_3, x_4, x_5]] = \\ [[x_1, x_2, x_3], \alpha_1(x_4), \alpha_2(x_5)] + [\alpha_1(x_3), [x_1, x_2, x_4], \alpha_2(x_5)] \\ +[\alpha_1(x_3), \alpha_2(x_4), [x_1, x_2, x_5]].\end{aligned}$$

n-ary Hom-Nambu algebras

Theorem. Let (V, m) be an n-ary Nambu algebra and let $\rho : V \to V$ be an n-ary Nambu algebras endomorphism.
$m_\rho = \rho \circ m$
$\widetilde{\rho} = (\rho, \cdots, \rho)$.
Then $(V, m_\rho, \widetilde{\rho})$ is an n-ary Hom-Nambu algebra.

n-ary Hom-Nambu-Lie algebras

Definition

A ternary Hom-Nambu algebra $\big(V, [\cdot,\cdot,\cdot], (\alpha_1, \alpha_2)\big)$ is called a *ternary Hom-Nambu-Lie algebra* if the bracket is skew-symmetric, that is

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = Sgn(\sigma)[x_1, x_2, x_3]$$

$\forall\, \sigma \in \mathcal{S}_3$ and $\forall\, x_1, x_2, x_3 \in V$

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

J. Arnlind, N. Makhlouf, S. Silvestrov, Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras, Journal of Mathematical Physics 51, 1, 2010

Definition

$(V, [\cdot,\cdot])$ binary algebra
$\tau : V \to \mathbb{K}$ linear map.
Define ternary bracket (trilinear map) $[\cdot,\cdot,\cdot]_\tau : V \times V \times V \to V$:

$$[x, y, z]_\tau = \tau(x)[y, z] + \tau(y)[z, x] + \tau(z)[x, y]. \tag{3}$$

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

If the bilinear multiplication $[\cdot, \cdot]$ in Definition 3 is skew-symmetric, then the trilinear map $[\cdot, \cdot, \cdot]_\tau$ is skew-symmetric as well.

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

If τ is a linear function such that $\tau([x, y]) = 0$ for all $x, y \in V$, then we call τ a *trace function on* $(V, [\cdot, \cdot])$. It follows immediately that $\tau([x, y, z]_\tau) = 0$ for all $x, y, z \in V$ if τ is a trace function.

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Theorem

$(V, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra and $\beta : V \to \mathbb{K}$ be a linear map. Assume that τ is a trace function on V fulfilling

$$\tau\big(\alpha(x)\big)\tau(y) = \tau(x)\tau\big(\alpha(y)\big) \tag{4}$$

$$\tau\big(\beta(x)\big)\tau(y) = \tau(x)\tau\big(\beta(y)\big) \tag{5}$$

$$\tau\big(\alpha(x)\big)\beta(y) = \tau\big(\beta(x)\big)\alpha(y) \tag{6}$$

for all $x, y \in V$.
Then $(V, [\cdot,\cdot,\cdot]_\tau, (\alpha, \beta))$ is a Hom-Nambu-Lie algebra.

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

If we choose $\beta = \alpha$ conditions for trace τ reduce to

$$\tau(\alpha(x))\tau(y) = \tau(x)\tau(\alpha(y)).$$

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Example

V vector space of $n \times n$ matrices
$\alpha(x) = s^{-1}xs$ for an invertible matrix s
Then $(V, \alpha \circ [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra. For matrices, any trace function is proportional to the matrix trace, so we let $\tau(x) = \mathrm{tr}(x)$. If we want to choose a $\beta \neq 0$, it can be proved that β has to be proportional to α, i.e. $\beta = \lambda\alpha$ for some $\lambda \neq 0$. Since $\mathrm{tr}\left(\alpha(x)\right) = \mathrm{tr}(x)$ it is clear that $(\alpha, \lambda\alpha, \mathrm{tr})$ is a nondegenerate compatible triple on V, which implies that $\left(V, [\cdot,\cdot,\cdot]_{\mathrm{tr}}, (\alpha, \lambda\alpha)\right)$ is a Hom-Nambu-Lie algebra induced from $(V, \alpha \circ [\cdot,\cdot], \alpha)$.

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Example

Let us start with the vector space V spanned by $\{x_1, x_2, x_3, x_4\}$ with a skew-symmetric bilinear map defined through

$$[x_i, x_j] = a_{ij}x_3 + b_{ij}x_4$$

where a_{ij} and b_{ij} are antisymmetric 4×4 matrices. Defining

$$\begin{aligned} &\alpha(x_i) = x_3 \qquad \beta(x_i) = x_4 \qquad i = 1, \dots, 4 \\ &\tau(x_1) = \gamma_1 \qquad \tau(x_2) = \gamma_2 \qquad \tau(x_3) = \tau(x_4) = 0, \end{aligned}$$

one immediately observes that τ is a trace function, $\operatorname{im}\alpha \subseteq \ker\tau$, $\operatorname{im}\beta \subseteq \ker\tau$, and $\beta \neq \alpha$.

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Example cont.
Furthermore, $(V, [\cdot,\cdot], \alpha)$ is a Hom-Lie algebra provided

$$b_{13} = b_{12} + b_{23}$$
$$b_{14} = b_{12} + b_{23} + b_{34}$$
$$b_{24} = b_{23} + b_{34}.$$

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Example cont.

By introducing $a = b_{12}$, $b = b_{23}$ and $c = b_{34}$, the four independent ternary brackets of the induced Hom-Nambu-Lie algebra can be written as

$$[x_1, x_2, x_3] = (\gamma_1 a_{23} - \gamma_2 a_{13})x_3 + (\gamma_1 b - \gamma_2(a+b))x_4$$
$$[x_1, x_2, x_4] = (\gamma_1 a_{24} - \gamma_2 a_{14})x_3 + (\gamma_1(b+c) - \gamma_2(a+b+c))x_4$$
$$[x_1, x_3, x_4] = (\gamma_1 a_{34})x_3 + (\gamma_1 c)x_4$$
$$[x_2, x_3, x_4] = (\gamma_2 a_{34})x_3 + (\gamma_2 c)x_4.$$

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Example cont.
For instance, choosing $\gamma_1 = \gamma_2 = 1$ and $a_{i<j} = 1$, one obtains the Hom-Nambu-Lie algebra

$$(\langle x_1, x_2, x_3, x_4\rangle, [\cdot,\cdot,\cdot], (\alpha,\beta))$$

defined by

$$\begin{aligned}
[x_1, x_2, x_3] &= -ax_4 \\
[x_1, x_2, x_4] &= -cx_4 \\
[x_1, x_3, x_4] &= x_3 + cx_4 \\
[x_2, x_3, x_4] &= x_3 + cx_4
\end{aligned}$$

together with $\alpha(x_i) = x_3$ and $\beta(x_i) = x_4$.

Some references

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AB – BA

Thank you for the music,

the songs I'm singing!

Thanks for all the joy

they're bringing!