Hilbert's basis theorem for hom-associative Ore extensions

Johan Richter

Blekinge Institute of Technology

Describes mostly joint work with Per Bäck. Johan Öinert, Patrik Nystedt and Sergei Silvestrov have contributed to papers cited.

Rings are unital

All rings in this talk are unital, i.e. there is an element 1 such that $1a = a1 = a$. Will try to mention every time we assume it is associative.

Ore extensions, motivation

Introduced by Norwegian mathematician Øystein Ore, under the name of *noncommutative polynomial rings.*

Take an associative ring R and consider the additive group $R[x]$. Want to give it a new multiplication.

Ore extensions, motivation

Would like $R[x]$ to be an associative ring. Would also like $\deg(ab) = \deg(a) + \deg(b)$ or at least $\deg(ab) \leq \deg(a) + \deg(b)$. Would also like $x^n \cdot x^m = x^{n+m}$.

If $r \in R$ we must have $xr = \sigma(r)x + \delta(r)$, for some functions σ and δ.

In general we must have

$$ax^m \cdot bx^n = \sum_{i \in \mathbb{N}} a\pi_i^m(b)x^{i+n}, \tag{1}$$

for $a, b \in R$ and $m, n \in \mathbb{N}$, where π_i^m denotes the sum of all the $\binom{m}{i}$ possible compositions of i copies of σ and $m - i$ copies of δ in arbitrary order.

Conditions on σ and δ

Want the Ore extension to be an associative ring.

$$x(r+s) = xr + xs.$$

$$x(rs) = (xr)s.$$

Conditions on σ

σ has to satisfy:

- $\sigma(1) = 1$;
- $\sigma(a + b) = \sigma(a) + \sigma(b)$;
- $\sigma(ab) = \sigma(a)\sigma(b)$.

So σ is an endomorphism.

Conditions on δ

δ must satisfy:

- $\delta(a + b) = \delta(a) + \delta(b)$;
- $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$.

A δ satisfying this is called a σ-derivation.

A ring

For σ and δ satisfying above conditions we get a ring $R[x; \sigma, \delta]$, called an Ore extension.

Degree

Can measure the *degree* of elements in an Ore extension in the same way as in the polynomial ring. Eg $\deg(x^2 - 3x) = 2$.

$$\deg(ab) = \deg(a) + \deg(b)$$

if σ injective and R does not contain zero-divisors.

Examples

Example

If $\sigma = \mathrm{id}_R$ and $\delta = 0$ then $R[x; \sigma, \delta]$ is isomorphic to $R[x]$, the polynomial ring in one central indeterminate.

Example

If $\sigma = \mathrm{id}_R$ then $R[x; \mathrm{id}_R, \delta]$ is a ring of differential polynomials.

Example

If $\delta = 0$ then $R[x; \sigma, 0]$ is a skew polynomial ring.

Examples II

Example

Take $R = k[y]$, $\sigma(p(y)) = p(qy)$, where $q \in k \setminus \{0, 1\}$ and $\delta(y) = q$. Then $R[x; \sigma, \delta]$ is called the q-Weyl algebra.

Example

Take $R = k[y]$, $\sigma = \mathrm{id}$ and $\delta(y) = 1$. Then $R[x; \sigma, \delta]$ is the ordinary Weyl algebra.

Hilbert basis theorem

Theorem

If R is an associative, Noetherian ring, σ is an automorphism of R and δ is a σ-derivation then $R[x;\sigma,\delta]$ is a also Noetherian.

As background to hom-associative Ore extensions we will describe non-associative Ore extensions.

Non-associative rings

By a non-associative ring we mean a not necessarily associative ring. Must be unital and distributive.

Construction

Let σ and δ be additive maps such that $\sigma(1) = 1$ and $\delta(1) = 0$. As before we equip $R[X]$ with a new multiplication.

The ring structure on $R[X; \sigma, \delta]$ is defined on monomials by

$$aX^m \cdot bX^n = \sum_{i \in \mathbb{N}} a\pi_i^m(b)X^{i+n}, \tag{2}$$

for $a, b \in R$ and $m, n \in \mathbb{N}$, where π_i^m denotes the sum of all the $\binom{m}{i}$ possible compositions of i copies of σ and $m - i$ copies of δ in arbitrary order.

Theorem (Nystedt, Öinert and R.)

Suppose that R is a non-associative ring and that δ right or left linear over the constants. If we put $D = R[X; \mathrm{id}_R, \delta]$, then the following assertions hold:

(a) *If R is δ-simple, then every ideal of D is generated by a unique monic polynomial in $Z(D)$;*

(b) *If R is δ-simple, then there is a monic $b \in R_\delta[X]$, unique up to addition of an element $k \in Z(R)_\delta$, such that $Z(D) = Z(R)_\delta[b]$;*

(c) *D is simple if and only if R is δ-simple and $Z(D)$ is a field. In that case $Z(D) = Z(R)_\delta$ in which case $b = 1$;*

(d) *If R is δ-simple, δ is a derivation on R and $\mathrm{char}(R) = 0$, then either $b = 1$ or there is $c \in R_\delta$ such that $b = c + X$. In the latter case, $\delta = \delta_c$;*

(e) *If R is δ-simple, δ is a derivation on R and $\mathrm{char}(R) = p > 0$, then either $b = 1$ or there is $c \in R_\delta$ and $b_0, \ldots, b_n \in Z(R)_\delta$, with $b_n = 1$, such that $b = c + \sum_{i=0}^{n} b_i X^{p^i}$. In the latter case, $\sum_{i=0}^{n} b_i \delta^{p^i} = \delta_c$.*

Further generalization

Nystedt, Öinert and Richter have further generalized non-associative Ore extension to monoid Ore extensions [6].

Hom-associative rings

A hom-associative ring (R, α) is a non-associative ring R together with an additive function $\alpha : R \to R$ such that

$$\alpha(a)(bc) = (ab)\alpha(c)$$

for all $a, b, c \in R$.
Introduced by Makhlouf and Silvestrov. Connection with Hom-Lie algebras which were defined earlier.
Include the non-associative rings as a special case with $\alpha \equiv 0$.

Hom-associative Ore extensions

Hom-associative Ore extensions were introduced by Bäck, Richter and Silvestrov.
Can ask when non-associative Ore extension can be made into Hom-associative. Some messy conditions in general. Nice special case.
If (R, α) is a hom-associative ring, σ is an endomorphism, δ is a σ-derivation, and α commutes with σ and δ then $(R[x; \sigma, \delta], \alpha)$ is hom-associative, where α has been extended as $\alpha(\sum x_i x^i) = \sum \alpha(a_i) x^i$.

Non-unitality

Unitality is perhaps not so natural when studying hom-associative rings so our definition of hom-associative Ore extensions is actually for non-unital rings. However for purposes of this talk it is the unital case that is interesting.

Hom-ideals

In hom-associative ideals we are interested in hom-ideals, which are ideals that are invariant under the twisting map α.
Using hom-ideals we can define hom-Noetherian rings in the obvious way.

Hilbert basis theorem

Theorem (Bäck and R.)

Let $\alpha\colon R \to R$ be the twisting map of a unital, hom-associative ring R, and extend the map homogeneously to $R[X;\sigma,\delta]$. Assume further that α commutes with δ and σ, and that σ is an automorphism and δ a σ-derivation on R. If R is right (left) hom-noetherian, then so is $R[X;\sigma,\delta]$.

Non-associative Hilbert basis theorem

Corollary

Let R be a unital, non-associative ring, σ an automorphism and δ a σ-derivation on R. If R is right (left) noetherian, then so is $R[X;\sigma,\delta]$.

Quaternion example

Given a unital and associative algebra A with product $\cdot$ over a field of characteristic different from two, one may define a unital and non-associative algebra A^+ by using the *Jordan product* $\{\cdot,\cdot\}\colon A^+ \to A^+$ given by $\{a,b\} := \frac{1}{2}(a\cdot b + b\cdot a)$ for any $a,b \in A$. A^+ is then a *Jordan algebra*, i.e. a commutative algebra where any two elements a and b satisfy the *Jordan identity*, $\{\{a,b\}\{a,a\}\} = \{a,\{b,\{a,a\}\}\}$.

Example

Let σ be the automorphism on $\mathbb{H}$ defined by $\sigma(i) = -i$, $\sigma(j) = k$, and $\sigma(k) = j$. Any automorphism on $\mathbb{H}$ is also an automorphism on $\mathbb{H}^+$, and hence $\mathbb{H}^+[X;\sigma,0_{\mathbb{H}}]$ is a unital, non-associative Ore extension where e.g. $X\cdot i = -iX$, $X\cdot j = kX$, and $X\cdot k = jX$. $\mathbb{H}^+[X;\sigma,0_{\mathbb{H}}]$ is then noetherian.

Example

Let $[j, \cdot]_{\mathbb{H}}$ be the inner derivation on $\mathbb{H}$ induced by j. Any derivation on $\mathbb{H}$ is also a derivation on $\mathbb{H}^+$, and so we may form the unital, non-associative Ore extension $\mathbb{H}^+ [X; \mathrm{id}_{\mathbb{H}}, [j, \cdot]_{\mathbb{H}}]$ which is noetherian due to 7. Here, $X \cdot i = iX - 2k$, $X \cdot j = jX$, and $X \cdot k = kX + 2i$.

Octonic example

Example

For R any non-associative ring, the *non-associative Weyl algebra over* R is the iterated, unital, non-associative Ore extension $R[Y][X; \mathrm{id}_R, \delta]$ where $\delta\colon R[Y] \to R[Y]$ is an R-linear map such that $\delta(1) = 0$. Considering $\mathbb{O}$ as a ring, the unital, non-associative Ore extension of $\mathbb{O}$ in the indeterminate Y is the unital and non-associative polynomial ring $\mathbb{O}[Y; \mathrm{id}_{\mathbb{O}}, 0_{\mathbb{O}}]$, for which we write $\mathbb{O}[Y]$. Let $\delta\colon \mathbb{O}[Y] \to \mathbb{O}[Y]$ be the $\mathbb{O}$-linear map defined on monomials by $\delta\left(aY^m\right) = maY^{m-1}$ for arbitrary $a \in \mathbb{O}$ and $m \in \mathbb{N}$, with the interpretation that $0aY^{-1}$ is 0. One readily verifies that δ is an $\mathbb{O}$-linear derivation on $\mathbb{O}[Y]$, and $\delta(1) = 0$. We thus define the *Weyl algebra over the octonions*, or the *octonionic Weyl algebra*, as $\mathbb{O}[Y][X; \mathrm{id}_{\mathbb{O}[Y]}, \delta]$ where δ is said derivation. Hence, in $\mathbb{O}[Y][X; \mathrm{id}_{\mathbb{O}[Y]}, \delta]$, $X \cdot Y = YX - 1$. The octonionic Weyl algebra is noetherian.

Hom-example

Example

This example is adapted from an example in [3].
Let K be a field and let $A = K[Y]$. Let U be a finite-dimensional K-algebra that is not associative, for example the non-abelian two-dimensional Lie algebra. We make U into an A-module by defining $au = a_0 u$ for $a \in A, u \in U$, where a_0 is the constant term of a. Any K-morphism of U is automatically also an A-morphism. Set $B = A \times U$ and define a multiplication on B by

$$(a_1, u_1) \cdot (a_2, u_2) = (a_1 a_2, a_1 u_2 + a_2 u_1 + u_1 u_2).$$

Further set $\alpha(a, u) = (Ya, Yu) = (Ya, 0)$. Then B is a not associative, hom-associative algebra with twisting map α. It is clearly noetherian.

Continued

Let β be any automorphism of U and define

$$\sigma(a, u) = (a, \beta(u)).$$

Then we claim that σ is an automorphism of B and furthermore it commutes with α.
So $B[X; \sigma, 0]$ is a hom-noetherian, hom-associative Ore extension.

Continued

We verify hom-associativity as follows:

$\alpha(a_1, u_1) \cdot ((a_2, u_2) \cdot (a_3, u_3)) = (Ya_1, 0) \cdot (a_2 a_3, a_2 u_3 + a_3 u_2 + u_2 u_3) = (Ya_1 a_2 a_3, 0)$.

$((a_1, u_1) \cdot (a_2, u_2)) \cdot \alpha(a_3, u_3) = (a_1 a_2, a_1 u_2 + a_2 u_1 + u_1 u_2) \cdot (Ya_3, 0) = (Ya_1 a_2 a_3, 0)$

Noetherian

Suppose we have an ascending chain of left ideals, $I_1 \subseteq I_2 \subseteq \ldots$, in B. (The case of right ideals is dealt with in an analogous fashion.) Define $J_j = \{a \in A \mid \exists u \in U : (a, u) \in I_j\}$. This is an ideal in A. Also define $H_j = \{u \in U \mid (0, u) \in I_j\}$. This is a left algebra ideal in U. We thus have two ascending chains, $J_1 \subseteq J_2 \subseteq \ldots$ and $H_1 \subseteq H_2 \subseteq \ldots$, in A and U, respectively. Since A and U are noetherian there is some integer n such that if $k > n$ then $J_k = J_n$ and $H_k = H_n$. We claim that in fact also $I_k = I_n$. Let $(a, u) \in I_k$. Then $a \in J_k = J_n$ so there is $v \in U$ such that $(a, v) \in I_n$. It follows that $u - v \in H_k = H_n$, which implies that $(0, u - v) \in I_n$. Hence $(a, u) = (a, v) + (0, u - v)$ is a sum of two elements in I_n and therefore belongs to I_n.

References

- P. Bäck, J. Richter and S. Silvestrov, Hom-associative Ore extensions and weak unitalizations, Int. Electron. J. Algebra **24** (2018), pp. 174–194. *arXiv*: 1710.04190

- P. Bäck and J. Richter, *Hilbert's basis theorem for non-associative and hom-associative Ore extensions*, arXiv:1804.11304

- Y. Frégier, A. Gohr, *On unitality conditions for hom-associative algebras*, arXiv:0904.4874

- A. Makhlouf, S. D. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2** (2008), pp. 51–64.

- P. Nystedt, J. Öinert and J. Richter, Non-associative Ore extensions, Isr. J. Math. (2018) *arXiv*:1509.01436

- P. Nystedt, J. Öinert, J. Richter, *Simplicity of Ore monoid rings*, J. Algebra **530** (2019), pp. 69–85 *arXiv*:1705.02778