Hom-Lie structures on 3-dimensional skew symmetric algebras

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2nd Swedish Network in Algebra and Geometry Workshop

BTH, 2019

October 18, 2019

Overview

Introduction

The study of Hom-Lie algebras is attributed to Hartwig, Larsson, and Silvestrov. This involves an introduction of a new linear map (twist) to the Lie algebra case.

Definition

[2] A Hom-Lie algebra is a triple $(V, [\cdot,\cdot], \alpha)$ consisting of a linear space V, bilinear map $[\cdot,\cdot] : V \times V \longrightarrow V$ and a linear space homomorphism $\alpha : V \to V$ satisfying

$$[x, y] = -[y, x] \qquad (skew - symettry) \tag{1}$$

$$\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0 \qquad (Hom - Jacobi\ identity) \tag{2}$$

for all $x, y, z \in V$, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z.

Hom-Lie structures

Hom-Lie structures, denoted by HomLie(μ), is the space of all linear endomorphisms that satisfy the Hom-Jacobi identity for some skew symmetric algebra L.

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That is the vector subspace

$$HomLie(\mu) = \{\alpha \in End\ V | \circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0,\ x, y, z \in L\}.$$

Equations of structure constants

Let $\{e_1, e_2, \ldots, e_n\}$ be basis for V.

The structure constant equations are given by;

$$[e_i, e_j] = \sum_{s=1}^{n} C_{ij}^{s} e_s \tag{3}$$

and

$$\alpha(e_i) = \sum_{t=1}^{n} a_{it} e_t \tag{4}$$

Replacing 3 and 4 in the Hom-Jacobi identity, we have a system of polynomial equations;

$$\sum_{s,t=1}^{n} a_{it} C_{jk}^{s} C_{ts}^{r} + a_{jt} C_{ki}^{s} C_{ts}^{r} + a_{kt} C_{ij}^{s} C_{ts}^{r} = 0, \qquad r = 1, 2, \ldots, n. \qquad (5)$$

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Writing the equations as linear in $a'_{it}s$, we rewrite 5 as;

$$\sum_{t=1}^{n} \{ a_{it} \left(\sum_{s=1}^{n} C_{jk}^{s} C_{ts}^{r} \right) + a_{jt} \left(\sum_{s=1}^{n} C_{ki}^{s} C_{ts}^{r} \right) + a_{kt} \left(\sum_{s=1}^{n} C_{ij}^{s} C_{ts}^{r} \right) \} = 0, \quad (6)$$

$$1 \leq i < j < k \leq n, \;\; r = 1, 2, \ldots, n.$$

Thus 6 becomes

$$Ma_\alpha = 0$$

where column matrix a_α is;

$$a_\alpha = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \\ a_{12} \\ \vdots \\ a_{n2} \\ \vdots \\ \vdots \\ a_{nn} \end{pmatrix}$$

M is a matrix with n^2 columns and $\binom{n}{3} \cdot n$ rows.

3−Dimensional Hom Lie algebras

In three dimension Hom-Lie algebras the system of polynomial equations is obtained from 5. Together with the skew- symmetry condition, the equation becomes;

$$\sum_{\substack{m,n=1 \\ m<n}}^{3} \Big(\circlearrowleft_{i,j,k} \left(a_{im} C_{jk}^{n} C_{mn}^{r} - a_{in} C_{jk}^{m} C_{mn}^{r} \right) \Big) = 0 \qquad (7)$$

for $r = 1, 2, 3$ and $1 \leq i < j < k \leq 3$, where the symbol $\circlearrowleft_{i,j,k}$ denotes a summation over the cyclic permutation on i, j, k.

Matrix M is a 3 by 9 matrix given as;

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} & M_{1,4} & M_{1,5} & M_{1,6} & M_{1,7} & M_{1,8} & M_{1,9} \\ M_{2,1} & M_{2,2} & M_{2,3} & M_{2,4} & M_{2,5} & M_{2,6} & M_{2,7} & M_{2,8} & M_{2,9} \\ M_{3,1} & M_{3,2} & M_{3,3} & M_{3,4} & M_{3,5} & M_{3,6} & M_{3,7} & M_{3,8} & M_{3,9} \end{pmatrix}$$

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$$M_{r,1} = (C_{23}^2 C_{12}^r + C_{23}^3 C_{13}^r)$$
$$M_{r,2} = (-C_{13}^2 C_{12}^r - C_{13}^3 C_{13}^r)$$
$$M_{r,3} = (C_{12}^2 C_{12}^r + C_{12}^3 C_{13}^r)$$
$$M_{r,4} = (C_{23}^3 C_{23}^r - C_{23}^1 C_{12}^r)$$
$$M_{r,5} = (C_{13}^1 C_{12}^r - C_{13}^3 C_{23}^r)$$
$$M_{r,6} = (C_{12}^3 C_{23}^r - C_{12}^1 C_{12}^r)$$
$$M_{r,7} = (-C_{23}^1 C_{13}^r - C_{23}^2 C_{23}^r)$$
$$M_{r,8} = (C_{13}^1 C_{13}^r + C_{13}^2 C_{23}^r)$$
$$M_{r,9} = (-C_{12}^1 C_{13}^r - C_{12}^2 C_{23}^r)$$

for $r = 1, 2, 3$.

Hom-Lie Structures Dimension

M represents a linear map $L : \mathbb{K}^9 \to \mathbb{K}^3$, and hence $a_\alpha = \begin{pmatrix} a_{11} & a_{21} & a_{31} & a_{12} & a_{22} & a_{32} & a_{13} & a_{23} & a_{33} \end{pmatrix}^t$ must be in the $\ker(L)$ for us to realize a Hom-Lie algebra.

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Rank $(M)+\dim(\ker(L)) = 9$ (nullity-rank theorem)

$$\implies 6 \leq \dim HomLie(\mu) \leq 9$$

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Let C the matrix of structure constants given as

$$C = \begin{pmatrix} C_{12}^1 & C_{12}^2 & C_{12}^3 \\ C_{13}^1 & C_{13}^2 & C_{13}^3 \\ C_{23}^1 & C_{23}^2 & C_{23}^3 \end{pmatrix}$$

Rank $C = 3$

Proposition

Let L be a 3-dimensional skew symmetric algebra with structure constants $\{C_{ij}^k\}_{i<j}$ and C be the matrix of structure constants. The dimension of the space of possible endomorphisms, HomLie(μ), attains minimum dimension 6, if and only if $\det C$ *is non-zero.*

Proof.

For Rank $(M) = 3$, then there exists a non-zero sub-determinant of a 3×3 sub-matrix of M.

From the computations all the non-zero sub-determinants have $\det C$ as a factor.

□

For example;

$$C^3_{23}(C^2_{13}C^3_{23} - C^3_{13}C^2_{23})(\det C)$$

$$-C^3_{13}(C^2_{13}C^3_{23} - C^3_{13}C^2_{23})(\det C)$$

$$C^3_{12}(C^2_{13}C^3_{23} - C^3_{13}C^2_{23})(\det C)$$

$$-C^2_{23}(C^2_{13}C^3_{23} - C^3_{13}C^2_{23})(\det C)$$

$$C^2_{13}(C^2_{13}C^3_{23} - C^3_{13}C^2_{23})(\det C)$$

$$\vdots$$

Rank $C = 2$

Proposition

Let L be a 3-dimensional skew symmetric algebra with structure constants $\{C_{ij}^k\}_{i<j}$ and C be the matrix of structure constants. If rank C is 2 then the dimension of the space of possible endomorphisms, HomLie(μ), is 7.

Proof.

From previous results, we see that if C is not of full rank then dim HomLie$(\mu) \geq 7$. From computations, if rank $(C) = 2$, there exists a non-zero sub-determinant of a 2×2 sub-matrix of M.

We investigate all possible cases for rank $C = 2$ (linear dependency of rows and/or zero rows) and compare in the list of the 2×2 sub-determinants of M. For each case, there exists at least one such non-zero subdeterminant. □

Rank $C = 1$, with dim HomLie$(\mu) = 8$

From the results it follows that if rank C is 1 then the dimension of HomLie$(\mu) > 7$.

However, if rank $C = 1$ this dimension can either be 8 or 9.

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dim HomLie$(\mu) = 8$

$$\mathfrak{C}'_1 = \begin{pmatrix} C^1_{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C^1_{12} \neq 0 \qquad \mathfrak{C}'_2 = \begin{pmatrix} 0 & C^2_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C^2_{12} \neq 0$$

$$\mathfrak{C}'_3 = \begin{pmatrix} 0 & 0 & 0 \\ C^1_{13} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C^1_{13} \neq 0 \qquad \mathfrak{C}'_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C^3_{13} \\ 0 & 0 & 0 \end{pmatrix}, C^3_{13} \neq 0$$

$$\mathfrak{C}'_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C^2_{23} & 0 \end{pmatrix}, C^2_{23} \neq 0 \qquad \mathfrak{C}'_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C^3_{23} \end{pmatrix}, C^3_{23} \neq 0$$

Rank $C = 1$, with dim HomLie$(\mu) = 9$

$$\mathfrak{C}_1 = \begin{pmatrix} C_{23}^3 & C_{12}^2 & -\frac{C_{12}^2 \ C_{23}^3}{C_{13}^1} \\ C_{13}^1 & \frac{C_{12}^2 \ C_{13}^1}{C_{23}^3} & -C_{12}^2 \\ -\frac{C_{13}^1 \ C_{23}^3}{C_{12}^2} & -C_{13}^1 & C_{23}^3 \end{pmatrix} C_{12}^2, C_{13}^1, C_{23}^3 \neq 0$$

Rank $C = 1$, with dim HomLie$(\mu) = 9$

$$\mathfrak{C}_1 = \begin{pmatrix} C_{23}^3 & C_{12}^2 & -\dfrac{C_{12}^2\ C_{23}^3}{C_{13}^1} \\ C_{13}^1 & \dfrac{C_{12}^2\ C_{13}^1}{C_{23}^3} & -C_{12}^2 \\ -\dfrac{C_{13}^1\ C_{23}^3}{C_{12}^2} & -C_{13}^1 & C_{23}^3 \end{pmatrix} C_{12}^2, C_{13}^1, C_{23}^3 \neq 0$$

$$\mathfrak{C}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{23}^1 & 0 & 0 \end{pmatrix}, C_{23}^1 \neq 0 \qquad \mathfrak{C}_3 = \begin{pmatrix} C_{23}^3 & 0 & C_{12}^3 \\ 0 & 0 & 0 \\ \dfrac{(C_{23}^3)^2}{C_{12}^3} & 0 & C_{23}^3 \end{pmatrix}, C_{12}^3 \neq 0$$

Rank $C = 1$, with dim HomLie$(\mu) = 9$

$$\mathfrak{C}_4 = \begin{pmatrix} 0 & 0 & 0 \\ C_{13}^1 & C_{13}^2 & 0 \\ \dfrac{-(C_{13}^1)^2}{C_{13}^2} & -C_{13}^1 & 0 \end{pmatrix}, C_{13}^2 \neq 0$$

$$\mathfrak{C}_5 = \begin{pmatrix} 0 & C_{12}^2 & C_{12}^3 \\ 0 & \dfrac{(C_{12}^2)^2}{C_{12}^3} & -C_{12}^2 \\ 0 & 0 & 0 \end{pmatrix}, C_{12}^3 \neq 0$$

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Proposition

If two skew-symmetric algebras (L, μ) and (L', μ') are isomorphic, then HomLie(μ) is isomorphic to HomLie(μ').

Proof.

Let $\phi : L \to L'$ be the isomorphism of the algebras. Let $\alpha \in \text{HomLie}(\mu)$ and $\beta \in \text{HomLie}(\mu')$. The isomorphism between the Hom-Lie structures $\varphi : HomLie(\mu) \to HomLie(\mu')$ is defined by $\beta = \varphi(\alpha) := \phi \circ \alpha \circ \phi^{-1}$. Given $x, y, z \in L$, then there exists $x', y', z' \in L'$ as images under the isomorphism ϕ, that is $\phi(x) = x', \phi(y) = y'$ and $\phi(z) = z'$. $\beta = \phi \circ \alpha \circ \phi^{-1}$ implies $\beta(x')) = \phi(\alpha(x)), \beta(y')) = \phi(\alpha(y))$ and $\beta(z')) = \phi(\alpha(z))$.

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We show that if $\alpha \in \text{HomLie}(\mu)$ then $\beta \in \text{HomLie}(\mu')$.

$$\begin{aligned}
&\mu'(\beta(x'), \mu'(y', z')) + \mu'(\beta(y'), \mu(z', x')) + \mu'(\beta(z'), \mu'(x', y')) \\
&= \mu'(\phi(\alpha(x)), \mu'(\phi(y), \phi(z)) + \mu'(\phi(\alpha(y)), \mu(\phi(z), \phi(x)) \\
&\quad + \mu'(\phi(\alpha(z)), \mu'(\phi(x), \phi(y)) \\
&= \phi[\mu((\alpha(x)), \mu(y, z)) + \mu((\alpha(y)), \mu(z, x)) + \mu((\alpha(z)), \mu(x, y))] \\
&= 0
\end{aligned}$$

□

Some Examples

Example

The Heisenberg Lie algebra: $[e_1, e_2] = e_3,\ [e_1, e_3] = 0,\ [e_2, e_3] = 0$

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies \dim \mathrm{HomLie}(\mu) = 9.$$

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The Lie algebra:

$[f_1, f_2] = -f_1 - f_2 + f_3, \ [f_1, f_3] = -f_1 - f_2 + f_3, \ [f_2, f_3] = f_1 + f_2 - f_3$

is isomorphic to the Heisenberg Lie algebra. Here,

$$C = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \implies \dim \mathrm{HomLie}(\mu) = 9.$$

These algebras are found in classes $\mathfrak{C}_3$ and $\mathfrak{C}_1$ respectively.

Example

Simple Lie algebra, $\mathfrak{sl}(2,\mathbb{C})$: $[e_1, e_2] = -2e_1,\ [e_1, e_3] = e_2,\ [e_2, e_3] = -2e_3$

$$C = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \implies \dim \mathrm{HomLie}(\mu) = 6.$$

Example

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$$C = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \implies \dim \mathrm{HomLie}(\mu) = 6.$$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{13} & a_{22} & a_{31} \\ 2a_{12} & a_{32} & a_{22} \end{pmatrix}, \text{ with 6 parameters } a_{11}, a_{12}, a_{13}, a_{13}, a_{22}, a_{32} \in \mathbb{C}.$$

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Thank you!