Natural generalizations of Kaplansky's conjectures

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BTH

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Reference

Units, zero-divisors and idempotents in rings graded by torsion-free groups, arXiv:1904.04847 [math.RA]

Outline

The group ring $A[G]$

- Ingredients: a unital ring A and a group G
- $A[G]$ is a free left (and right) A-module with G as its basis; Every element is of the form $a_1g_1 + a_2g_2 + \ldots + a_ng_n$.
- Multiplication is defined by

$$(a_1g_1)(a_2g_2) = (a_1a_2)(g_1g_2)$$

for $a_1, a_2 \in A$ and $g_1, g_2 \in G$.

Remark

$A[G]$ has a natural G-gradation!
Put $R_g := Ag$ for $g \in G$. Then

- $A[G] = R = \oplus_{g \in G} R_g$, and
- $R_g R_h \subseteq R_{gh}$, for all $g, h \in G$.

In fact, we even have $R_g R_h = R_{gh}$ in this case.

Kaplansky's conjectures for group rings

Kaplansky 1956, Smirnov & Bovdi 1968, Kaplansky 1970.

Problem

Let $\mathbb{F}$ be a field, let G be a torsion-free group and denote by $\mathbb{F}[G]$ the corresponding group ring.

(a) *Is every unit in $\mathbb{F}[G]$ necessarily trivial, i.e. of the form ag for some $a \in \mathbb{F}$ and $g \in G$?*

(b) *Is $\mathbb{F}[G]$ necessarily a domain?*

(c) *Is every idempotent in $\mathbb{F}[G]$ necessarily trivial, i.e. either 0 or 1?*

Remark

Torsion-freeness is a necessary condition for $\mathbb{F}[G]$ to be a domain. Consider the equality $(g^n - 1) = (g - 1)(g^{n-1} + g^{n-2} + \ldots + 1)$.

A set of generalized problems...

Remember:

- $\mathbb{F}[G]$ is a strongly G-graded ring. In particular, its gradation is non-degenerate.
- In particular, $(\mathbb{F}[G])_e = \mathbb{F}$ is a domain.

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Let G be a torsion-free group and let R be a unital G-graded ring equipped with a non-degenerate G-gradation such that R_e is a domain.

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Let G be a torsion-free group and let R be a unital G-graded ring equipped with a non-degenerate G-gradation such that R_e is a domain.

(a) *Is every unit in R necessarily homogeneous w.r.t. the given G-gradation?*

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(a) *Is every unit in R necessarily homogeneous w.r.t. the given G-gradation?*

(b) *Is R necessarily a domain?*

(c) *Is every idempotent in R necessarily trivial?*

Unique product groups and two unique products groups

Definition

Let G be a group.

(a) G is said to be a *unique product group* if, given any two non-empty finite subsets X and Y of G, there exists at least one element $g \in G$ which has a unique representation of the form $g = xy$ with $x \in X$ and $y \in Y$.

(b) G is said to be a *two unique products group* if, given any two non-empty finite subsets X and Y of G with $|X| + |Y| > 2$, there exist at least two distinct elements g and h of G which have unique representations of the form $g = xy$, $h = x'y'$ with $x, x' \in X$ and $y, y' \in Y$.

Unique product groups, continued

Lemma (Strojnowski, 1980)

A group G is a unique product group if and only if it is a two unique products group.

Remark

- Every unique product group is necessarily torsion-free.
- A torsion-free group is **not necessarily** a unique product group.

Example (Unique product groups)

Right (or left) orderable groups, including e.g.

- all torsion-free abelian groups,
- all torsion-free nilpotent groups,
- all free groups.

A solution for unique product groups

Theorem

Let G be a unique product group and let R be a unital G-graded ring whose G-gradation is fully component regular. The following three assertions hold:

(i) *Every unit in R is homogeneous;*

(ii) *R is a domain;*

(iii) *Every idempotent in R is trivial.*

Proof.

(ii) Suppose that $ab = 0$ for some non-zero $a, b \in R$. Consider the finite subsets $X = \mathrm{Supp}(a)$ and $Y = \mathrm{Supp}(b)$. There is some $g \in XY$ which may be written $g = xy$ in a unique way. But then $ab = 0$ implies $a_x b_y = 0$. This is a contradiction. □

Primeness

Theorem

Let G be a torsion-free group and let R be a G-graded ring. If the G-gradation on R is non-degenerate and R_e is a domain, then R is a prime ring.

Proof.

Rather technical. □

Nilpotents vs. zero-divisors

Proposition

Let G be a torsion-free group and let R be a G-graded ring whose G-gradation is non-degenerate. Then R is a domain if and only if R is reduced and R_e is a domain.

Proof.

- If R is a domain, then R is reduced.
- Suppose that R_e is a domain and that R is not a domain. Notice that R is prime. Choose non-zero $a, b \in 0$ such that $ab = 0$. By primeness we have $bRa \neq \{0\}$. Notice that

$$(bRa)^2 \subseteq bRabRa = \{0\}.$$

Thus, there is a non-zero $r \in R$ with $r^2 = 0$.

□

A hierarchy

Theorem

Let G be a torsion-free group and let R be a unital G-graded ring. Furthermore, suppose that the G-gradation on R is non-degenerate and that R_e is a domain. Consider the following assertions:

(i) *Every unit in R is homogeneous;*

(ii) *R is reduced;*

(iii) *R is a domain;*

(iv) *Every idempotent in R is trivial.*

Then $\mathrm{(i)}\Rightarrow\mathrm{(ii)}\Rightarrow\mathrm{(iii)}\Rightarrow\mathrm{(iv)}$. *Moreover,* $\mathrm{(iii)}\Rightarrow\mathrm{(ii)}$.

Proof.

(i)⇒(ii) Suppose that $a \in R$ satisfies $a^2 = 0$. Then $(1_R + a)(1_R - a) = (1_R - a)(1_R + a) = 1_R$ and thus $1_R - a \in R_g$ for some $g \in G$. One can show that $a \in R_{\langle g \rangle}$ and that $a = 0$. □

Central elements

Theorem

Let G be a torsion-free group and let R be a unital G-graded ring. If the G-gradation on R is non-degenerate and R_e is a domain, then the following three assertions hold:

(i) *Every central unit in R is homogeneous;*

(ii) *R has no non-trivial central zero-divisor;*

(iii) *Every central idempotent in R is trivial.*

Proof.

Based on the following observations:

- If $c \in R$ is a central element, then the subgroup of G generated by $\mathrm{Supp}(c)$ is an FC-group.
- Every torsion-free FC-group is abelian (credit: Neumann).

□

Generalized conjectures

Conjecture

Let G be a torsion-free group and let R be a unital G-graded ring whose G-gradation is non-degenerate. If R_e is a domain, then the following assertions hold:

(a) Every unit in R is homogeneous;

(b) R is a domain;

(c) Every idempotent in R is trivial.

Generalized conjectures

Conjecture

Let G be a **finitely generated** torsion-free group and let R be a unital G-graded ring whose G-gradation is non-degenerate. If R_e is a domain, then the following assertions hold:

(a) Every unit in R is homogeneous;

(b) R is a domain;

(c) Every idempotent in R is trivial.

Examples beyond group rings and crossed products

Example (The 1st Weyl algebra)

Let $\mathbb{F}$ be a field and consider *the 1st Weyl algebra* $R = \mathbb{F}\langle x, y\rangle/(yx - xy - 1)$.

- By assigning suitable degrees to the generators, $\deg(x) = 1$ and $\deg(y) = -1$, R becomes graded by the unique product group $(\mathbb{Z}, +)$.
- $R_0 = \mathbb{F}[xy]$ is an integral domain.
- The $\mathbb{Z}$-gradation is non-degenerate.

By our results, the first Weyl algebra R is a domain.

Remark

The above gradation is **not** strong.

Examples beyond group rings and crossed products, cont'd

More generally:

Example (Crystalline graded ring)

Every *crystalline graded ring* $R = A \diamond_\sigma^\alpha G$ is equipped with a non-degenerate G-gradation with $R_e = A$.
If A is a domain and G is a unique product group, then R is a domain by our results.

Remark

If a crystalline graded ring is strongly G-graded, then it is a G-crossed product.

More examples ...?

Question

Is there an example of a ring which is strongly graded by a torsion-free group, but which is not a crossed product?

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Is there an example of a ring which is strongly graded by a torsion-free group, but which is not a crossed product?

Yes!

Example (Leavitt path algebra)

Consider a Leavitt path algebra $R = L_K(E)$ where E is a finite directed graph which looks like the number 8. (Not a cycle, no sources, no sinks.)

- R is strongly $\mathbb{Z}$-graded (hence also non-degenerate).
- R is not a $\mathbb{Z}$-crossed product.

Remark

BUT... R_0 is not a domain since $v(v-1) = 0$ for any vertex $v \in E^0$.

More examples ...?

Question

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Is there an example of a domain which is strongly graded by a torsion-free group, but which is not a crossed product?

For the moment, I cannot answer this question!

Example (A possible candidate)

What about strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings?

The end

THANK YOU FOR YOUR ATTENTION!