Object unital groupoid graded rings

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Outline

- **Noether (1930)**
- **Question (2004)**
- **Unital groupoid graded rings**
- **Question (2019)**
- **Object unital groupoid graded rings**

Definition (Noether 1930)

Let L/K denote a finite Galois field extension with Galois group G (identity e). This gives us a map

$$G \ni g \mapsto \alpha_g \in \mathrm{Aut}_K(L)$$

Suppose that

$$G \times G \ni (g,h) \mapsto \beta_{g,h} \in L \setminus \{0\}$$

is a Noether factor set, that is

$$\forall g,h,p \in G \quad \beta_{g,e} = \beta_{e,g} = 1 \quad \beta_{g,h}\beta_{gh,p} = \alpha_g(\beta_{h,p})\beta_{g,hp}$$

The crossed product algebra $L \rtimes_{\beta}^{\alpha} G$ is defined to be the set of formal sums of the form $\sum_{g \in G} l_g u_g$ where $l_g \in L$, for $g \in G$. If $\sum_{g \in G} l_g u_g$ and $\sum_{g \in G} l'_g u_g$ are two such formal sums, then their sum is defined to be

$$\sum_{g \in G} l_g u_g + \sum_{g \in G} l'_g u_g = \sum_{g \in G} (l_g + l'_g) u_g.$$

The product of two such formal sums is defined to be the additive extension of the relations

$$l_g u_g \cdot l'_h u_h = l_g \alpha_g(l'_h) \beta_{g,h} u_{gh}$$

for $g, h \in G$.

Theorem

$L \rtimes_\beta^\alpha G$ **is an associative and simple** K**-algebra.**

Example

If $L = \mathbb{C}$**,** $K = \mathbb{R}$**,** $G = C_2 = \{1, g\}$ **the cyclic group of order two,** $\alpha_g : \mathbb{C} \to \mathbb{C}$ **is conjugation, and** $\beta_{g,g} = -1$**, then** $L \rtimes_\beta^\alpha G = \mathbb{H} =$ **Hamilton's quaternions.**

Remark

If we put $R = L \rtimes_{\beta}^{\alpha} G$ **and** $R_g = Lu_g$**, for** $g \in G$**, then**

$$R = \oplus_{g \in G} R_g$$

and

$$R_g R_h \subseteq R_{gh}$$

so R **is a ring graded by the group** G**. In fact**

$$R_g R_h = R_{gh}$$

so that R **is strongly graded.**

Question (2004)

The definition of $L \rtimes_{\beta}^{\alpha} G$ does not make sense if L/K is a finite separable (but not necessarily normal) field extension. Can we modify the definition of $L \rtimes_{\beta}^{\alpha} G$ so that it makes sense in this more general situation?

One solution is to consider groupoid graded rings.

Definition

By a groupoid $\mathcal{G}$ we mean a small category with the property that all morphisms are isomorphisms. Equivalently, it can be defined by saying that $\mathcal{G}$ is a non-empty set equipped with a unary operation

$$\mathcal{G} \ni \sigma \mapsto \sigma^{-1} \in \mathcal{G} \quad \textbf{(inversion)}$$

and a partial binary operation

$$\mathcal{G} \times \mathcal{G} \ni (\sigma, \tau) \mapsto \sigma\tau \in \mathcal{G} \quad \textbf{(composition)}$$

such that $\forall \sigma, \tau, \rho \in \mathcal{G}$ the following four axioms hold:

- $(\sigma^{-1})^{-1} = \sigma$

- **if $\sigma\tau$ and $\tau\rho$ are defined, then $(\sigma\tau)\rho$ and $\sigma(\tau\rho)$ are defined and equal**

- **the domain $d(\sigma) := \sigma^{-1}\sigma$ is always defined and if $\sigma\tau$ is defined, then $d(\sigma)\tau = \tau$**

- **the range $r(\tau) := \tau\tau^{-1}$ is always defined and if $\sigma\tau$ is defined, then $\sigma r(\tau) = \sigma$.**

$\mathcal{G}_0 := d(\mathcal{G}) = r(\mathcal{G})$ **is called the unit space of** $\mathcal{G}$

$\mathcal{G}_1 := \mathcal{G}$

$\mathcal{G}_2 := \{(\sigma, \tau) \in \mathcal{G} \times \mathcal{G} \mid \sigma\tau$ **is defined**$\}$

$\mathcal{G}_3 := \{(\sigma, \tau, \rho) \in \mathcal{G} \times \mathcal{G} \times \mathcal{G} \mid \sigma\tau$ **and** $\tau\rho$ **are defined**$\}$

Background

Groupoids were introduced by Brandt in 1926 in relation to his work on generalizing to quaternary quadratic forms a composition of binary quadratic forms (due to Gauss).

Brandt later used groupoids in generalizing to the non-commutative case the arithmetic of ideals in rings of algebraic integers, replacing the classical finite group by a finite groupoid.

Brandt's axioms for groupoids influenced Eilenberg and Mac Lane in their definition of a category.

Example

All groups G are groupoids with $G_0 = \{e\}$ where e is the identity element of G.

In fact, a groupoid $\mathcal{G}$ is a group $\iff |\mathcal{G}_0| = 1$.

Example

A disjoint union

$$\mathcal{G} = \biguplus_{i \in I} G_i$$

of groups G_i is a groupoid.

The product gh is defined precisely when g and h belong to the same group.

We can even form disjoint unions of groupoids.

Example

Let X be a non-empty set and R an equivalence relation on X. The composition $(a, b)(c, d)$ is defined precisely when $b = c$ and is then equal to (a, d). The inverse of (a, b) is (b, a).

Special case 1: $R = X \times X$

Special case 2: $R = \Delta = \{(x, x) \mid x \in X\}$ **"="** X

Example

Let G be a group and let X be a set equipped with a left G-action $G \ni g \mapsto (\alpha_g : X \to X)$. The action groupoid $G \ltimes X$ is the set $G \times X$ equipped with partial composition $(h, y)(g, x) = (hg, x)$ defined provided $y = \alpha_g(x)$.

So X, G and $G \times X$ are all groupoids on their own!

Example

The fundamental groupoid $\pi_1(X)$ of a topological space X. Elements are homotopy classes $[\gamma]$ of continuous maps $\gamma : [0, 1] \to X$.

Example

The transformations of the children's game fifteen puzzle form a groupoid (not a group, as not all moves can be composed).

Definition

Suppose that R is a ring and that $\mathcal{G}$ is a groupoid. We say that R is graded by $\mathcal{G}$ if there for all $\sigma \in \mathcal{G}$ is an additive subgroup R_σ of R such that

$$R = \oplus_{\sigma \in \mathcal{G}} R_\sigma$$

and $\forall \sigma, \tau \in \mathcal{G}$

$$R_\sigma R_\tau \subseteq R_{\sigma\tau} \quad \text{if} \quad (\sigma, \tau) \in G_2$$

$$R_\sigma R_\tau = \{0\} \quad \text{if} \quad (\sigma, \tau) \notin G_2.$$

In that case, if $\forall (\sigma, \tau) \in G_2 \quad R_\sigma R_\tau = R_{\sigma\tau}$**, then we say that R is strongly graded by $\mathcal{G}$.**

Remark

If K is a field, then $M_n(K)$ is graded by $\mathcal{G} = I \times I$ if we put $M_n(K)_{(i,j)} = Ke_{i,j}$ where $e_{i,j}$ denotes the $n \times n$ matrix with 1 in the ijth position and 0 elsewhere.

In other words:

$$M_n(K) = \oplus_{(i,j)\in\mathcal{G}} M_n(K)_{(i,j)}$$

$$M_n(K)_{(i,j)} M_n(K)_{(k,l)} = \delta_{j,k} M_n(K)_{(i,l)}$$

Therefore $M_n(K)$ is strongly $\mathcal{G}$-graded.

Definition (2004)

Let L/K be a finite separable field extension. Let $\overline{K}$ denote a fixed algebraic closure of K containing L. Choose a normal closure N/K of L/K in $\overline{K}$. Put

$$G = \mathsf{Gal}(N/K)$$

and let

$$L_1, L_2, \cdots, L_n$$

denote the different conjugate fields of L under the action of G. For all $i, j \in I$ we put

$$G_{ij} = \{g|_{L_j} \mid g \in G \text{ and } g(L_j) = L_i\}$$

and let $\mathcal{G} = \biguplus_{i,j \in I} G_{ij}$. Then $\mathcal{G}$ is a groupoid with partial composition $\forall i, j, k \quad G_{ij} \times G_{jk} \to G_{ik}$.

For all $\sigma \in \mathcal{G}_{ij}$ **put** $\alpha_\sigma = \sigma : L_j \to L_i$**. For all** $(\sigma, \tau) \in \mathcal{G}_2$ **let** $\beta_{\sigma,\tau} \in L_{r(\sigma)} \setminus \{0\}$ **satisfy**

$$\beta_{\sigma,d(\sigma)} = \beta_{r(\sigma),\sigma} = 1_{L_{r(\sigma)}}$$

and

$$\beta_{\sigma,\tau}\beta_{\sigma\tau,\rho} = \alpha_\sigma(\beta_{\tau,\rho})\beta_{\sigma,\tau\rho}$$

for all $(\sigma, \tau, \rho) \in \mathcal{G}_3$**.**

The crossed product algebra $L \rtimes_{\beta}^{\alpha} \mathcal{G}$ is defined to be the set of formal sums of the form $\sum_{\sigma \in \mathcal{G}} l_\sigma u_\sigma$ where $l_\sigma \in L_{r(\sigma)}$, for $\sigma \in \mathcal{G}$. If $\sum_{\sigma \in \mathcal{G}} l_\sigma u_\sigma$ and $\sum_{\sigma \in \mathcal{G}} l'_\sigma u_\sigma$ are two such formal sums, then their sum is defined to be

$$\sum_{\sigma \in \mathcal{G}} l_\sigma u_\sigma + \sum_{\sigma \in \mathcal{G}} l'_\sigma u_\sigma = \sum_{\sigma \in \mathcal{G}} (l_\sigma + l'_\sigma) u_\sigma.$$

The product of two such formal sums is defined to be the additive extension of the relations

$$l_\sigma u_\sigma \cdot l'_\tau u_\tau = l_\sigma \alpha_\sigma(l'_\tau) \beta_{\sigma,\tau} u_{\sigma\tau}$$

for $(\sigma, \tau) \in \mathcal{G}_2$, and $l_\sigma u_\sigma \cdot l'_\tau u_\tau = 0$ for $(\sigma, \tau) \in \mathcal{G} \times \mathcal{G}$ with $(\sigma, \tau) \notin \mathcal{G}_2$.

Theorem (2004)

- $L \rtimes_{\beta}^{\alpha} \mathcal{G}$ **is an associative and simple** K**-algebra.**

- $L \rtimes_{\beta}^{\alpha} \mathcal{G}$ **is unital with**

$$1_{L \rtimes_{\beta}^{\alpha} \mathcal{G}} = \sum_{i=1}^{n} 1_{L_i} u_{\mathrm{id}_{L_i}}$$

- $L \rtimes_{\beta}^{\alpha} \mathcal{G}$ **is strongly graded.**

Proposition (2004)

If R is a unital ring which is graded by a groupoid $\mathcal{G}$, then there is a subgroupoid $\mathcal{H}$ of $\mathcal{G}$ such that

- $R = \oplus_{\sigma \in \mathcal{H}} R_\sigma$;
- **for all $e \in \mathcal{H}_0$ the ring R_e is non-zero and unital;**
- $\mathcal{H}_0$ **is finite;**
- $1_R = \sum_{e \in \mathcal{H}_0} 1_{R_e}$.

Definition

Suppose that $\mathcal{G}$ is a groupoid. A non-empty subset $\mathcal{H}$ of $\mathcal{G}$ is called a subgroupoid of $\mathcal{G}$ if

- $h \in \mathcal{H} \Rightarrow h^{-1} \in \mathcal{H}$**, and**

- $g, h \in \mathcal{H} \cap \mathcal{G}_2 \Rightarrow gh \in \mathcal{H}$**.**

In that case, $\mathcal{H}$ is called a wide subgroupoid of $\mathcal{G}$ if $\mathcal{H}_0 = \mathcal{G}_0$.

Question (2019)

Can we define $L \rtimes_{\beta}^{\alpha} \mathcal{G}$ if L/K is a (not necessarily finite) separable (but not necessarily normal) field extension?

One solution is to consider non-unital groupoid graded rings.

Definition

If R is a ring which is graded by a groupoid $\mathcal{G}$, then we say that R is object unital if

- $\forall e \in \mathcal{G}_0 \quad R_e$ **is unital, and**

- $\forall \sigma \in \mathcal{G} \quad \forall r \in R_\sigma \quad 1_{R_{r(\sigma)}} r = r 1_{R_{d(\sigma)}} = r$.

Remark

If R is a ring which is graded by a groupoid $\mathcal{G}$, and R is object unital, then R is a ring with enough idempotents (namely the 1_{R_e} for $e \in \mathcal{G}_0$).

Remark

If R is a ring which is graded by a group G, then the following are equivalent:

- **R is unital;**
- **R is object unital.**

Definition

If S is a ring and M is a left (or right) S-module, then M is said to be unitary if

$$SM = M$$

(or $MS = M$).

If S and T are rings and M is an S-T-bimodule, then M is said to be unitary if it is unitary both as a left S-module and as a right T-module.

Definition

Suppose that $\mathcal{G}$ is a groupoid and $E \subseteq \mathcal{G}_0$. We put

$$\mathcal{G}(E) = \{\sigma \in \mathcal{G} \mid d(\sigma) \in E \text{ and } r(\sigma) \in E\}.$$

It is easy to see that $\mathcal{G}(E)$ is a subgroupoid of $\mathcal{G}$. It is called the principal groupoid associated to E. If $E = \{e\}$, then $\mathcal{G}(E)$ is a group which is called the principal group associated to e.

Definition

Suppose that R is a ring which is graded by the groupoid $\mathcal{G}$. Let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$. We put

$$R_{\mathcal{H}} = \oplus_{\sigma \in \mathcal{H}} R_{\sigma}.$$

Then $R_{\mathcal{H}}$ is graded by $\mathcal{H}$.

Proposition

If R is a ring which is graded by a groupoid $\mathcal{G}$, then the following are equivalent:

- **R is object unital;**

- **for all $e \in \mathcal{G}_0$ the ring R_e is unital and for all $\sigma \in \mathcal{G}$ the $R_{r(\sigma)}$-$R_{d(\sigma)}$-bimodule R_σ is unitary;**

- **for all finite subsets E of $\mathcal{G}_0$, $R_{\mathcal{G}(E)}$ is unital.**

Proposition

Suppose that R is a ring which is graded by the groupoid $\mathcal{G}$. If R is object unital, then the following are equivalent:

- **R is strongly graded;**
- **for all finite subsets E of $\mathcal{G}_0$ the ring $R_{\mathcal{G}(E)}$ is strongly $\mathcal{G}(E)$-graded;**
- **for all $\sigma \in \mathcal{G}$ the equality $R_{r(\sigma)} = R_\sigma R_{\sigma^{-1}}$ holds;**
- **for all $\sigma \in \mathcal{G}$ the relation $1_{R_{r(\sigma)}} \in R_\sigma R_{\sigma^{-1}}$ holds.**

Crossed products?

Proposition

Suppose that R is a ring which is graded by the groupoid $\mathcal{G}$. If R is object unital and we put

$$\mathcal{G}' = \{\sigma \in G \mid 1_{R_{r(\sigma)}} \neq 0 \text{ and } 1_{R_{d(\sigma)}} \neq 0\},$$

then $\mathcal{G}'$ is a subgroupoid of $\mathcal{G}$ and $R = R_{\mathcal{G}'}$.

Remark

From now on we make the assumption that if R is object unital, then $\forall e \in \mathcal{G}_0 \quad 1_{R_e} \neq 0$.

Definition

Suppose that R is a ring which is graded by the groupoid $\mathcal{G}$. Suppose that R is object unital. An element $r \in R_\sigma$ is called object invertible if there is $s \in R_{\sigma^{-1}}$ such that

$$rs = 1_{R_{r(\sigma)}}$$

and

$$sr = 1_{R_{d(\sigma)}}.$$

We will refer to s as the object inverse of r (object inverses are unique).

Definition

Suppose that R is a ring which is graded by the groupoid $\mathcal{G}$. If $\sigma \in \mathcal{G}$ and $r \in R_\sigma \setminus \{0\}$, then we put

$$\deg(r) = \sigma$$

This defines a map

$$\deg : \bigcup_{\sigma \in \mathcal{G}} (R_\sigma \setminus \{0\}) \to \mathcal{G}$$

Definition

Suppose that R is a ring which is graded by the groupoid $\mathcal{G}$. Suppose that R is object unital. Put

$$U^{\text{gr}}(R) = \{\textbf{object invertible elements of } R\}.$$

We define a groupoid structure on $U^{\text{gr}}(R)$ in the following way. Take $u, v \in U^{\text{gr}}(R)$. The groupoid composition of u and v is defined and equal to

$$uv$$

precisely when $d(\text{deg}(u)) = r(\text{deg}(v))$. The groupoid inverse of u is the object inverse of u.

Definition

Suppose that R is a ring which is graded by the groupoid $\mathcal{G}$. Suppose that R is object unital. We say that R is an object crossed product if

$$\forall \sigma \in \mathcal{G} \quad U^{\mathrm{gr}}(R) \cap R_\sigma \neq \emptyset.$$

Remark

Object crossed products are strongly graded.

Definition

Suppose that $\mathcal{G}$ and $\mathcal{H}$ are groupoids. A map

$$f : \mathcal{G} \to \mathcal{H}$$

is said to be a groupoid homomorphism if for all $g, h \in \mathcal{G}$ with gh defined, $f(g)f(h)$ is also defined and

$$f(gh) = f(g)f(h).$$

In that case, f is said to be strong if for all $g, h \in \mathcal{G}$ with $f(g)f(h)$ defined, gh is also defined.

Definition

Suppose that $\mathcal{G}$ and $\mathcal{H}$ are groupoids and that

$$f : \mathcal{G} \to \mathcal{H}$$

is a groupoid homomorphism. The image of f is

$$\mathrm{Im}(f) = \{f(g) \mid g \in \mathcal{G}\}$$

and the kernel of f is

$$\mathrm{Ker}(f) = \{g \in \mathcal{G} \mid f(g) \in \mathcal{H}_0\}.$$

Proposition

If $\mathcal{G}$ and $\mathcal{H}$ are groupoids and

$$f : \mathcal{G} \to \mathcal{H}$$

is a groupoid homomorphism, then

- Ker(f) **is a subgroupoid of $\mathcal{G}$;**

- **if f is strong, then** Im(f) **is a subgroupoid of $\mathcal{H}$.**

Definition

Suppose that R is a ring which is graded by the groupoid $\mathcal{G}$. Suppose that R is object unital and let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$. Define a $\mathcal{G}$-grading on $R_{\mathcal{H}}$ in the following way. Take $\sigma \in \mathcal{G}$. Then put

$$(R_{\mathcal{H}})_\sigma = R_\sigma,$$

if $\sigma \in \mathcal{H}$, and

$$(R_{\mathcal{H}})_\sigma = \{0\},$$

otherwise.

Definition

Suppose that R is a ring which is graded by the groupoid $\mathcal{G}$. We put $R_0 = R_{\mathcal{G}_0}$ and consider R_0 as a $\mathcal{G}$-graded ring.

Proposition

Suppose that R is a ring which is graded by the groupoid $\mathcal{G}$. If R is object unital, then $U^{\mathrm{gr}}(R_0)$ equals the direct sum groupoid $\biguplus_{e\in\mathcal{G}_0} U(R_e)$ of the ordinary unit groups $\{U(R_e)\}_{e\in\mathcal{G}_0}$.

Proposition

Suppose that R is a ring which is graded by the groupoid $\mathcal{G}$. If R is object unital, then

$$\text{deg} : \bigcup_{\sigma \in \mathcal{G}} (R_\sigma \setminus \{0\}) \to \mathcal{G}$$

restricts to a strong groupoid homomorphism

$$\text{deg}_U : U^{\text{gr}}(R) \to \mathcal{G}$$

with kernel equal to $U^{\text{gr}}(R_0)$. The homomorphism deg_U **is surjective $\Leftrightarrow$ R is a crossed product.**

Examples of object crossed products?

Definition

Let $\mathcal{G}$ be a groupoid and suppose that we are given a collection

$$A = (A_e)_{e \in \mathcal{G}_0}$$

of unital rings A_e. For all $e, f \in \mathcal{G}_0$ let

$$\mathrm{Iso}_{e,f}(A) = \{\text{unital ring isomorphisms } A_f \to A_e\}$$

and put

$$\mathrm{Iso}(A) = \biguplus_{e,f \in \mathcal{G}_0} \mathrm{Iso}_{e,f}(A).$$

This is a groupoid with partial composition

$$\mathrm{Iso}_{e,f}(A) \times \mathrm{Iso}_{e',f'}(A) \to \mathrm{Iso}_{e,f'}(A)$$

defined precisely when $f = e'$.

By an object crossed system we mean a quadruple $(A, \mathcal{G}, \alpha, \beta)$ **where** $\alpha : \mathcal{G} \to \mathrm{Iso}(A)$ **and** $\beta : \mathcal{G}_2 \to U^{\mathrm{gr}}(A_0)$ **are maps satisfying the following four axioms for all** $(\sigma, \tau, \rho) \in \mathcal{G}_3$ **and all** $a \in A_{d(\tau)}$

(C1) $\alpha_\sigma : A_{d(\sigma)} \to A_{r(\sigma)}$ **and** $\alpha_e = \mathrm{id}_{A_e}$ **for** $e \in \mathcal{G}_0$

(C2) $\beta_{\sigma,\tau} \in U(A_{r(\sigma)})$ **and** $\beta_{\sigma,d(\sigma)} = \beta_{r(\sigma),\sigma} = 1_{A_{r(\sigma)}}$

(C3) $\alpha_\sigma(\alpha_\tau(a)) = \beta_{\sigma,\tau}\alpha_{\sigma\tau}(a)\beta_{\sigma,\tau}^{-1}$

(C4) $\beta_{\sigma,\tau}\beta_{\sigma\tau,\rho} = \alpha_\sigma(\beta_{\tau,\rho})\beta_{\sigma,\tau\rho}$**.**

The map α **is called a weak action of** $\mathcal{G}$ **on** A **and** β **is called an** α**-cocycle.**

Definition

Let $(A, \mathcal{G}, \alpha, \beta)$ be an object crossed system. Let $\{u_\sigma\}_{\sigma \in \mathcal{G}}$ be a copy of $\mathcal{G}$. By $A \rtimes_\beta^\alpha \mathcal{G}$ we mean the set of formal sums of the form

$$\sum_{\sigma \in \mathcal{G}} a_\sigma u_\sigma$$

where $a_\sigma \in A_{r(\sigma)}$, for $\sigma \in \mathcal{G}$, and $a_\sigma = 0$, for all but finitely many $\sigma \in \mathcal{G}$. If

$$\sum_{\sigma \in \mathcal{G}} a_\sigma u_\sigma \quad \text{and} \quad \sum_{\sigma \in \mathcal{G}} a'_\sigma u_\sigma$$

are two such sums, then their sum is defined to be

$$\sum_{\sigma \in \mathcal{G}} a_\sigma u_\sigma + \sum_{\sigma \in \mathcal{G}} a'_\sigma u_\sigma = \sum_{\sigma \in \mathcal{G}} (a_\sigma + a'_\sigma) u_\sigma.$$

The product of two such sums is defined to be the additive extension of the relations

$$a_\sigma u_\sigma \cdot a'_\tau u_\tau = a_\sigma \alpha_\sigma(a'_\tau)\beta_{\sigma,\tau} u_{\sigma\tau},$$

when $(\sigma, \tau) \in \mathcal{G}_2$**, and**

$$a_\sigma u_\sigma \cdot a'_\tau u_\tau = 0,$$

otherwise. For all $\sigma \in \mathcal{G}$ **we put**

$$(A \rtimes_\beta^\alpha \mathcal{G})_\sigma = A_{r(\sigma)} u_\sigma.$$

It is clear that this defines a $\mathcal{G}$**-grading on** $A \rtimes_\beta^\alpha \mathcal{G}$**.**

Proposition

If $(A, \mathcal{G}, \alpha, \beta)$ is an object crossed system, then $A \rtimes_{\beta}^{\alpha} \mathcal{G}$ is an object unital groupoid graded ring which is an object crossed product. Conversely, any object crossed product R can be presented in this way.

Definition

Let L/K **be a separable field extension. Let** $\overline{K}$ **be a fixed algebraic closure of** K **containing** L**. Choose a normal closure** N/K **of** L/K **in** $\overline{K}$**. Put**

$$G = \mathrm{Gal}(N/K)$$

and let

$$\tilde{L} = \{L_i\}_{i \in I}$$

denote the different conjugate fields of L **under the action of** G**. For all** $i, j \in I$ **we put**

$$G_{ij} = \{g|_{L_j} \mid g \in G \text{ and } g(L_j) = L_i\}$$

and let $\mathcal{G} = \biguplus_{i,j \in I} G_{ij}$**. Then** $\mathcal{G}$ **is a groupoid with partial composition** $\forall i, j, k \quad G_{ij} \times G_{jk} \to G_{ik}$**.**

Define

$$\alpha : \mathcal{G} \to \mathrm{Iso}(\tilde{L})$$

by $\alpha_\sigma = \sigma$ **for** $\sigma \in G_{ij}$**. If**

$$\beta : \mathcal{G}_2 \to U^{\mathrm{gr}}(\tilde{L}_0)$$

is a map satisfying conditions (C2) and (C4), then we say that $\tilde{L} \rtimes_\beta^\alpha \mathcal{G}$ **is the object crossed product defined by** L/K **and** β**.**

Proposition

$\tilde{L} \rtimes_\beta^\alpha \mathcal{G}$ **is a simple ring.**

Definition

If $(A, \mathcal{G}, \alpha, \beta)$ **is a crossed system with** β **trivial, that is if for all** $(\sigma, \tau) \in \mathcal{G}$ **the relation**

$$\beta_{\sigma,\tau} = 1_{A_{r(\sigma)}}$$

holds, then we call $A \rtimes^{\alpha}_{\beta} \mathcal{G}$ **an object skew groupoid ring and we denote it by**

$$A \rtimes^{\alpha} \mathcal{G}.$$

Proposition

If R is an object unital ring, then R is an object skew groupoid ring if and only if

$$\text{deg} : U^{\text{gr}}(R) \to \mathcal{G}$$

is surjective and split.

Definition

If $(A, \mathcal{G}, \alpha, \beta)$ is a crossed system with α trivial, that is if all the rings A_e, for $e \in \mathcal{G}_0$, coincide with the same ring B, and for all $\sigma \in \mathcal{G}$, the map $\alpha_\sigma : B \to B$ is the identity, then the corresponding object crossed product $A \rtimes_\beta^\alpha \mathcal{G}$ is called an object twisted groupoid ring and it is denoted by

$$B \rtimes_\beta \mathcal{G}.$$

In that case, $\beta_{\sigma,\tau} \in U(Z(B))$, for $(\sigma, \tau) \in \mathcal{G}_2$. This follows from (C3).

Proposition

If R is an object unital ring, then R is a twisted groupoid ring if and only if

- $\deg : U^{\mathrm{gr}}(R) \to \mathcal{G}$ **is surjective, and**

- **all the rings R_e, for $e \in \mathcal{G}_0$, are copies of the same ring B (the copy of an element $b \in B$ in R_e is denoted by b^e), and**

- **for all $\sigma \in \mathcal{G}$ there is an object invertible element $u_\sigma \in R_\sigma$ with the property that for all $b \in B$ the equality $b^{r(\sigma)} u_\sigma = u_\sigma b^{d(\sigma)}$ holds.**

Definition

If $(A, \mathcal{G}, \alpha, \beta)$ is a crossed system with α and β trivial, that is if all $(\sigma, \tau) \in \mathcal{G}$ the relation $\beta_{\sigma,\tau} = 1_{R_{r(\sigma)}}$ holds, and all the rings A_e, for $e \in \mathcal{G}_0$, coincide with the same ring B, and for all $\sigma \in \mathcal{G}$, the map $\alpha_\sigma : B \to B$ is the identity, then the corresponding object crossed product $A \rtimes_\beta^\alpha \mathcal{G}$ is called a groupoid ring and it is denoted by $B[\mathcal{G}]$.

Proposition

If R is an object unital ring, then R is a groupoid ring if and only if

- **the degree map $U^{\mathrm{gr}}(R) \to \mathcal{G}$ is split surjective, and**

- **all the rings R_e, for $e \in \mathcal{G}_0$, are copies of the same ring B (the copy of an element $b \in B$ in R_e is denoted by b^e), and**

- **for all $\sigma \in \mathcal{G}$ there is an object invertible element $u_\sigma \in R_\sigma$ with the property that for all $b \in B$ the equality $b^{r(\sigma)} u_\sigma = u_\sigma b^{d(\sigma)}$ holds.**

To be continued...

- **Cohomology theory ?**

- **Separability of $R_{\mathcal{G}}/R_{\mathcal{H}}$?**

- **The ungrading functor (direct summand, free, finitely generated, finitely presented, projective, injective, essential, small, flat ...) ?**

- **Crossed product algebras $L \rtimes_{\beta}^{\alpha} G$ defined by any field extension L/K (not necessarily finite or normal or separable or even algebraic) ?**

Thank you!