Minimal Embeddings in a Noncommutative Context

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Overview

Real Calculi, Definition

A real calculus is a structure $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$, where

- $\mathcal{A}$ is a unital *-algebra,
- $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ is a real Lie algebra of derivations on $\mathcal{A}$,
- M is a (right) $\mathcal{A}$-module, and
- $\varphi : \mathfrak{g} \to M$ is a $\mathbb{R}$-linear map such that $\varphi(\mathfrak{g})$ generates M.

Initial example: Let Σ be a smooth manifold. With

- $\mathcal{A} = \mathcal{C}^\infty(\Sigma)$,
- $\mathfrak{g} = \mathrm{Der}(\mathcal{C}^\infty(\Sigma))$,
- $M = \mathfrak{X}(\Sigma)$ (the module of smooth vector fields over Σ), and
- $\varphi =$ the natural isomorphism between smooth vector fields and derivations,

we have that $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ is a real calculus.

Metrics

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ be a real calculus. A *metric* $h : M \times M \to \mathcal{A}$ is a Hermitian form that is non-degenerate, i.e.

- $h(m_1 + m_2, n) = h(m_1, n) + h(m_2, n)$ for all $m_1, m_2, n \in M$,
- $h(m, n \cdot a) = h(m, n)a$ for all $m, n \in M$, $a \in \mathcal{A}$,
- $h(m, n) = h(n, m)^*$ for all $m, n \in M$, and
- $h(m, n) = 0$ for all $n \in M \Rightarrow m = 0$.

Moreover, if $h(\varphi(\partial_1), \varphi(\partial_2)) = h(\varphi(\partial_1), \varphi(\partial_2))^*$ for all $\partial_1, \partial_2 \in \mathfrak{g}$ (i.e., it is truly symmetric on $\varphi(\mathfrak{g})$) then $(C_{\mathcal{A}}, h)$ is called a real metric calculus.

Connections

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ be a real calculus. An affine connection $\nabla : \mathfrak{g} \times M \to M$ is a map that satisfies

- $\nabla_\partial(m + n) = \nabla_\partial m + \nabla_\partial n$ for all $m, n \in M$ and $\partial \in \mathfrak{g}$,
- $\nabla_{\lambda\partial_1+\partial_2} m = \lambda\nabla_{\partial_1} m + \nabla_{\partial_2} m$ for all $m \in M$, $\lambda \in \mathbb{R}$ and $\partial_1, \partial_2 \in \mathfrak{g}$, and
- $\nabla_\partial(m \cdot a) = (\nabla_\partial m) \cdot a + m \cdot \partial(a)$ for all $m \in M$, $\partial \in \mathfrak{g}$ and $a \in \mathcal{A}$.

With a metric, define a notion of a metric and torsion-free connection ∇ to satisfy the following:

$$\text{Metric:} \quad \partial(h(m, n)) = h(\nabla_\partial m, n) + h(m, \nabla_\partial n)\nabla_{\partial_i},$$
$$\text{Torsion-free:} \quad \nabla_{\partial_i}\varphi(\partial_j) - \nabla_{\partial_j}\varphi(\partial_i) - \varphi([\partial_i, \partial_j]) = 0.$$

If ∇ is metric and torsion-free, we say that $(C_{\mathcal{A}}, h, \nabla)$ is pseudo-Riemannian.

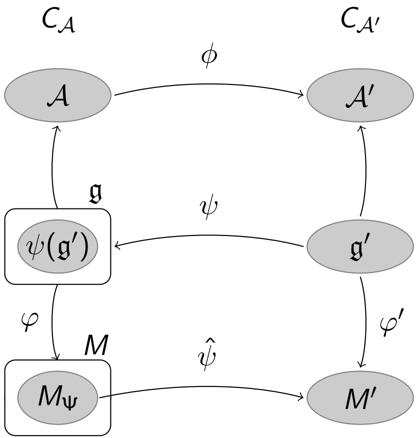
Real Calculus Homomorphisms

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ and $C_{\mathcal{A}'} = (\mathcal{A}', \mathfrak{g}', M', \varphi')$ be two real calculi. $(\phi, \psi, \hat{\psi})$ is said to be a real calculus homomorphism from $C_{\mathcal{A}}$ to $C_{\mathcal{A}'}$ if the following conditions are satisfied:

1. $\phi : \mathcal{A} \to \mathcal{A}'$ is a *-algebra homomorphism,
2. $\psi : \mathfrak{g}' \to \mathfrak{g}$ is a Lie homomorphism such that $\delta(\phi(a)) = \phi(\psi(\delta)(a))$ for all $\delta \in \mathfrak{g}'$ and $a \in \mathcal{A}$.
3. M_Ψ is the submodule of M generated by $\varphi(\psi(\mathfrak{g}'))$, and $\hat{\psi} : M_\Psi \to M'$ is a map that satisfies
   - $\hat{\psi}(m_1 + m_2) = \hat{\psi}(m_1) + \hat{\psi}(m_2)$ for all $m_1, m_2 \in M_\Psi$,
   - $\hat{\psi}(m \cdot a) = \hat{\psi}(m) \cdot \phi(a)$ for all $m \in M_\Psi$ and $a \in \mathcal{A}$, and
   - $\hat{\psi}(\varphi(\psi(\delta))) = \varphi'(\delta)$ for all $\delta \in \mathfrak{g}'$.

Illustration of a Real Calculus Homomorphism

A schematic picture illustrating the real calculus homomorphism $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$:



Embeddings of Real Calculi

A real calculus homomorphism $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is called an embedding of $C_{\mathcal{A}'}$ into $C_{\mathcal{A}}$ if ϕ is surjective and there is a submodule $\tilde{M}$ of M such that $M = M_{\Psi} \oplus \tilde{M}$.

Moreover, if $(C_{\mathcal{A}}, h)$ and $(C_{\mathcal{A}'}, h')$ are real metric calculi such that $h'(\hat{\psi}(m), \hat{\psi}(n)) = \phi(h(m, n))$ for all $m, n \in M_{\Psi}$ and $M = M_{\Psi} \oplus M_{\Psi}^{\perp}$ (w.r.t h), then we say that $(C_{\mathcal{A}'}, h)$ is isometrically embedded into $(C_{\mathcal{A}}, h)$ by $(\phi, \psi, \hat{\psi})$.

Orthogonal decomposition of ∇

Let $(C_{\mathcal{A}}, h, \nabla)$ and $(C_{\mathcal{A}'}, h', \nabla')$ be pseudo-Riemannian calculi such that $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ is an isometric embedding of $(C_{\mathcal{A}'}, h')$ into $(C_{\mathcal{A}}, h)$.

Let $m \in M_{\Psi}$ and let $\xi \in M_{\Psi}^{\perp}$. One may split ∇ into tangential and normal parts in the following way:

$$\nabla_{\psi(\delta)} m = L(\delta, m) + \alpha(\delta, m) \quad \text{(Gauss' formula)}$$
$$\nabla_{\psi(\delta)} \xi = -A_{\xi}(\delta) + D_{\delta}\xi \quad \text{(Weingarten's formula)};$$

$\alpha : \mathfrak{g}' \times M_{\Psi} \to M_{\Psi}^{\perp}$ is called the second fundamental form, and $A : \mathfrak{g}' \times M_{\Psi}^{\perp} \to M_{\Psi}$ is called the Weingarten map.

General results

- Proposition: L is the Levi-Civita connection for the embedded calculus.
- α and A behave like their classical counterparts:

$$\alpha(\delta_1, \Psi(\delta_2)) = \alpha(\delta_2, \Psi(\delta_1)),$$
$$\alpha(\lambda_1\delta_1 + \lambda_1\delta_2, m) = \lambda_1\alpha(\delta_1, m) + \lambda_2\alpha(\delta_2, m)$$
$$\alpha(\delta, m_1 a_1 + m_2 a_2) = \alpha(\delta_1, m_1)a_1 + \alpha(\delta_1, m_2)a_2,$$

and $h(A_\xi(\delta), m) = h(\xi, \alpha(\delta, m))$.

- Gauss' formula for the curvature of an embedding can be formulated and proven in the context of pseudo-Riemannian calculi:

$$\begin{aligned}\phi(h(E_1, R(\partial_3, \partial_4)E_2)) =& h'(E_1', R'(\delta_3, \delta_4)E_2') \\ &+ \phi(h(\alpha(\delta_4, E_1), \alpha(\delta_3, E_2))) \\ &- \phi(h(\alpha(\delta_3, E_1), \alpha(\delta_4, E_2)))\end{aligned}$$

Free real calculi

To study minimality of an embedding, we restrict our attention to so-called free real calculi. A real calculus $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$ is said to be free if M is free, and there is a basis $\{\partial_1, ..., \partial_k\}$ of $\mathfrak{g}$ that generates a basis $\{\varphi(\partial_1), ..., \varphi(\partial_k)\}$ of M.

Moreover, if $(C_{\mathcal{A}}, h)$ is a real metric calculus such that $C_{\mathcal{A}}$ is free and the metric h is invertible, then $(C_{\mathcal{A}}, h)$ is said to be a free real metric calculus.

- Far from all real metric calculi are also free, but one gains structure to work with in return.
- For a free real metric calculus $(C_{\mathcal{A}}, h)$, there is always a unique connection ∇ such that $(C_{\mathcal{A}}, h, \nabla)$ is pseudo-Riemannian.

Mean curvature and minimality of an embedding

Let $(C_{\mathcal{A}}, h, \nabla)$ and $(C_{\mathcal{A}'}, h', \nabla')$ be pseudo-Riemannian real calculi such that $(C_{\mathcal{A}}, h)$ and $(C_{\mathcal{A}'}, h')$ are free, and let $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ be an isometric embedding of $(C_{\mathcal{A}'}, h')$ into $(C_{\mathcal{A}}, h)$.

For a basis $\{\delta_1, ..., \delta_k\}$ of $\mathfrak{g}'$, define the mean curvature $H_{\mathcal{A}'} : M \to \mathcal{A}'$ as:

$$H_{\mathcal{A}'}(m) = \phi(h(m, \alpha(\delta_i, \Psi(\delta_j))))(h')^{ij}, \quad m \in M.$$

- The value of $H_{\mathcal{A}'}(m)$ is independent of the choice of basis $\{\delta_1, ..., \delta_k\}$ for all $m \in M$,
- $H_{\mathcal{A}'}(m) = 0$ for all $m \in M_\Psi$,
- We say that an embedding is minimal if the mean curvature is zero, i.e. $H_{\mathcal{A}'}(m) = 0$ for all $m \in M_\Psi^\perp$.

The noncommutative torus and the noncommutative 3-sphere

- T^2_θ : The noncommutative torus is the *-algebra with unital generators U, V satisfying the relation $VU = qUV$, where $q = e^{2\pi i\theta}$; Choose derivations δ_1, δ_2 given by:

$$\delta_1(U) = iU \qquad \delta_2(U) = 0$$
$$\delta_1(V) = 0 \qquad \delta_2(V) = iV.$$

  We have that $[\delta_1, \delta_2] = 0$.

- S^3_θ : The noncommutative 3-sphere is the unital *-algebra with generators Z, Z^*, W, W^* subject to the relations

$$WZ = qZW \qquad W^*Z = \bar{q}ZW^* \qquad WZ^* = \bar{q}Z^*W$$
$$W^*Z^* = qZ^*W^* \qquad Z^*Z = ZZ^* \qquad W^*W = WW^*$$
$$WW^* = \mathbb{1} - ZZ^*,$$

Continuation, and a concrete embedding of T_θ^2 into S_θ^3

Choose derivations $\partial_1, \partial_2, \partial_3$ of S_θ^3, given by:

$$\begin{aligned} &\partial_1(Z) = iZ, \qquad &&\partial_2(Z) = 0, \qquad &&\partial_3(Z) = Z|W|^2 \\ &\partial_1(W) = 0 \qquad &&\partial_2(W) = iW \qquad &&\partial_3(W) = -W|Z|^2, \end{aligned}$$

where $|Z|^2 := ZZ^*$ and $|W|^2 := WW^*$. We have that $[\partial_i, \partial_j] = 0$ for all $i, j = 1, 2, 3$.

An embedding of T_θ^2 into S_θ^3 is achieved by the *-homomorphism $\phi : S_\theta^3 \to T_\theta^2$ given by $\phi(Z) = \lambda U$ and $\phi(W) = \mu W$, where λ and μ are nonzero complex constants such that $|\lambda|^2 + |\mu|^2 = 1$. Embedding is minimal for the standard metric if $|\lambda| = |\mu| = 1/\sqrt{2}$.

Future research

- Future work on minimal embeddings involves trying to develop a more general definition of mean curvature dealing with cases where a module is, for instance, projective.
- More generally regarding real calculus homomorphisms, future work involves developing a better understanding of when two real calculi are isomorphic, and our current research is centered around classifying real calculi for finite noncommutative spaces.

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