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SEMILATTICE DECOMPOSITION OF SEMIGROUPS: FROM THE THEORY TO APPLICATIONS

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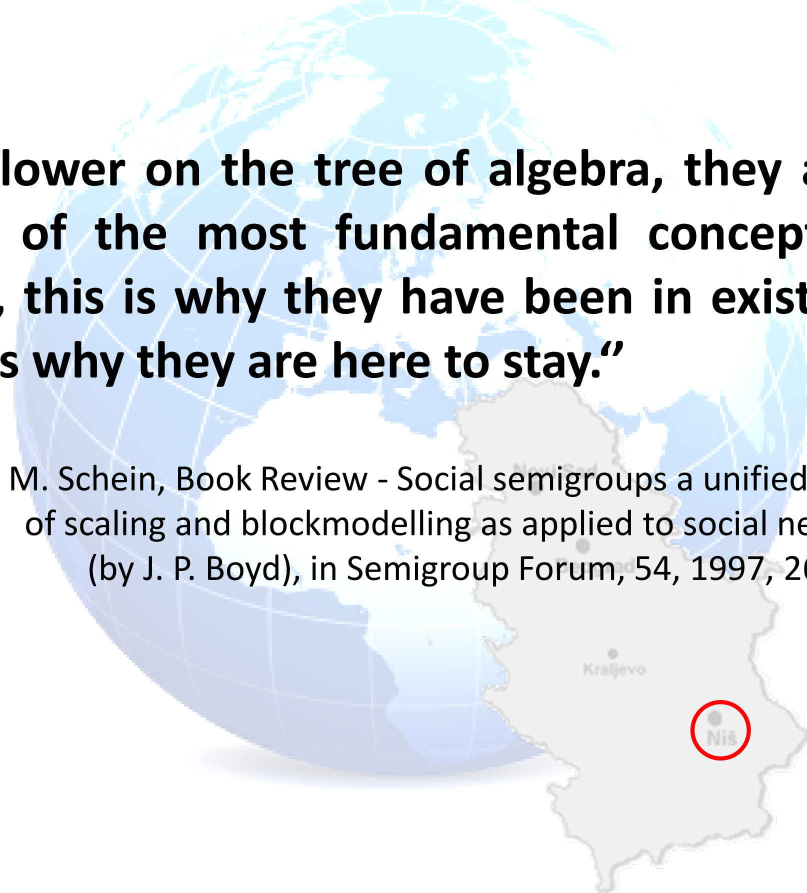
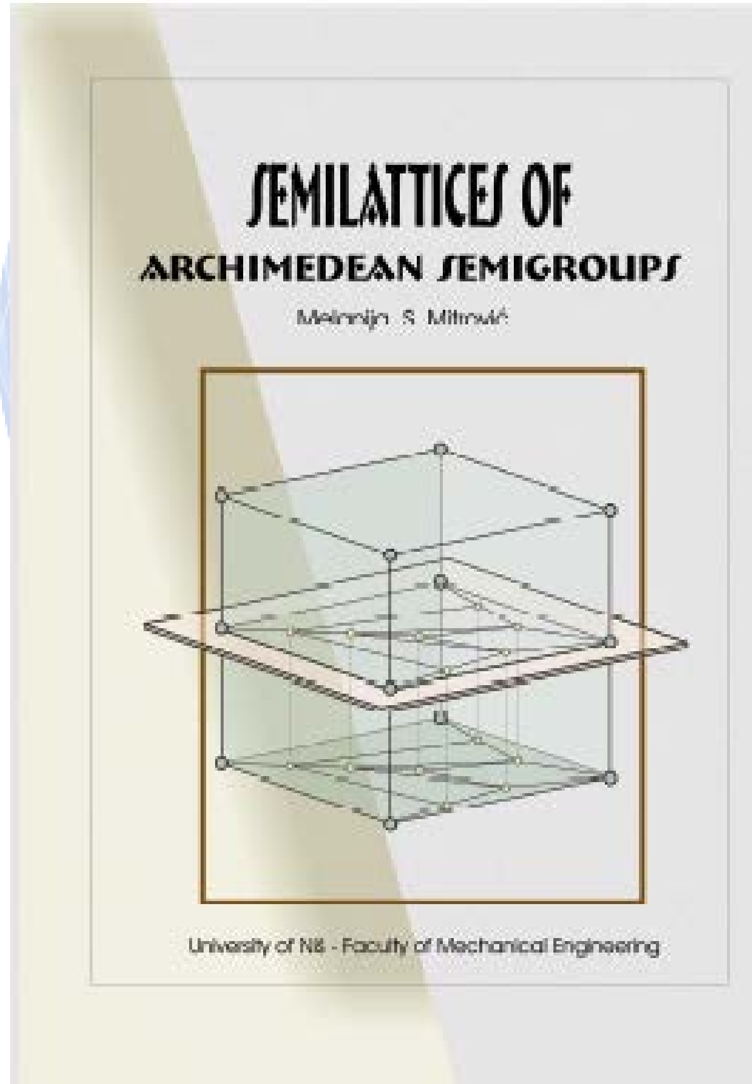
„Semigroups aren't a barren, sterile flower on the tree of algebra, they are a natural algebraic approach to some of the most fundamental concepts of algebra (and mathematics in general), this is why they have been in existence for more then half a century, and this is why they are here to stay.“

B. M. Schein, Book Review - Social semigroups a unified theory of scaling and blockmodelling as applied to social networks (by J. P. Boyd), in Semigroup Forum, 54, 1997, 264-268.



M. Mitrović, *Semilattices of Archimedean Semigroups*, University of Niš - Faculty of Mechanical Engineering, Niš (2003).

M. Mitrović, S. Silvestrov, *Semilattice decompositions of semigroups. Hereditarness and periodicity - an overview*, accepted in Stochastic Processes and Algebraic Structures - From Theory Towards Applications, Volume II: Algebraic Structures and Applications, (Eds.: S. Silvestrov, M. Malyarenko, M. Rančić), Springer, 2019.



Semigroup $(S, \cdot)$

set S together with an associative binary operation $\cdot$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \qquad \text{for any} \quad x, y, z \in S$$

- Where nature of the multiplications is clear from the context, it is written S rather than $(S, \cdot)$
- Frequently, xy is written rather than $x \cdot y$

1160 distinct of order 5
15 793 distinct of order 6
836 021 distinct of order 7
1 843 120 128 distinct of order 8
⋮

(neither isomorphic nor anti-isomorphic)

- Ring $(R, +, \cdot) \longrightarrow (R, \cdot)$ is a semigroup

OPEN QUESTION: conditions for a semigroup to be the multiplicative semigroup of a ring

- Green's relations, the fundamental tools in the structure theory of semigroups, **do not have nontrivial analogues** in groups, rings, quasigroups, lattices, universal algebras, fields.

Semigroups - development

In some sense semigroup theory started with a result on finite semigroups. More precisely, *A. K. Suschekewitsch's theorem* (1928) describes the structure of finite semigroups without proper ideals.

J. Almeida: Finite Semigroups and Universal Algebra, World Scientific, 1994.
P. A. Grillet: Semigroups - An Introduction to the Structure Theory, Marcel Dekker, Inc., 1995.

In the forties, with the works *of David Rees, James Alexander Green, Evgenii Sergeevich Ljapin, Alfred. H. Clifford, Gordon Preston,* the theory grew without giving particular emphasises to the finiteness of the semigroup.

C. Hollings: Mathematics Across the Iron Curtain: A History of the Algebraic Theory of Semigroups, Providence: American Mathematical Society, 2014.

Semigroup with a finiteness condition is a semigroup possesing any property which is valid for all finite semigroups.

Examples of finiteness conditions

- π-regularity,
- completely π-regularity,
- periodicity,
- finite generation,
- local finiteness,
- residual finiteness,
- finite presentation.

Applications of presented classes of semigroups and their semilattice decompositions in certain types of ring constructions, in particular in semigroup graded ring theory.

For a given semigroup S an associative ring R is said to be a *semigroup-graded ring* or, shortly, an *S-graded ring* if

$$R = \oplus_{s \in S} R_s$$

is a direct sum of additive subgroups R_s and

$$R_s R_t \subseteq R_{st}$$

for all $s, t \in S$.

"This is very pretty mathematics which illustrates the interplay between ring-theoretic and semigroup-theoretic techniques."

D. F. Anderson, Robert Gilmer's work on semigroup rings, in Multiplicative Ideal Theory in Commutative Algebra - A Tribute to the Work of Robert Gilmer - Editors: J. W. Brewer, S. Glaz, W. J. Heinzer, B. M. Olberding, Springer, 2006.

1 Basic concepts

2 Decomposition of a semigroup

3 Completely π-regular semigroups

4 Periodic semigroups

5 Concluding remarks

1. Basic concepts

- Special elements
- Special subsemigroups
- Different types of regularity

The study of all distinguished types of special elements is of interest in its own right, but, the results on these types of elements is often an important tool in the study of structure properties of semigroups.

1.1 Special elements and subsets

- A subset of S

$$\sqrt{A} = \{x \in S : (\exists n \in \mathbb{N})\, x^n \in A\}$$

- $\boldsymbol{E(S)} = \{a \in S \,|\, a = a^2\}$ (set of all idempotents of S)

S is a semigroup with idempotent(s): $E(S) \neq \emptyset$.

- S is *unipotent*: $\mid E(S) \mid = 1$.
  - S is *nil* semigroup: $S = \sqrt{\{0\}}$.
- S is a *band*: $\boldsymbol{S = E(S)}$.
  - S is a *semilattice*: commutative band.

S is *idempotent-free*: $E(S) \neq \emptyset$.

- S is *periodic*: $S = \sqrt{E(S)}$.

- $\boldsymbol{Reg(S)} = \{a \in S | (\exists x \in S)\, axa = a\}$

the regular part of S (set of all regular elements of S)

- $\boldsymbol{Gr(S)} = \{a \in S | (\exists x \in S)\, axa = a,\ ax = xa\}$

the group part of S (set of all completely regular or group element of S)

In general

$$\boldsymbol{E(S)} \subseteq \boldsymbol{Gr(S)} \subseteq \boldsymbol{Reg(S)} \subseteq \boldsymbol{S}.$$

1.3 Regular parts of subsemigroup

- T is a subset of S

$$E(T) = T \cap E(S)$$

- T is a subsemigroup of S

$$\boldsymbol{Reg(T)} \;=\; \{\, a \in T \mid (\exists x \in T)\; axa = a \,\}$$

regular part of T;

$$\boldsymbol{reg(T)} \;=\; \{\, a \in T \mid (\exists x \in S)\; axa = a \,\}$$

s-regular part of T.

In general

$$\boldsymbol{Reg(T) \subseteq reg(T) = T \cap Reg(S)}.$$

Problem: Semigroup S with

$$\boldsymbol{Reg(T) = reg(T)}$$

for every subsemigroup T of S.

$*\quad*\quad*$

– Equality (known up to the date):

T is two-sided ideal of S
local submonoid eSe, $e \in E(S)$, of S

1.5 Ideals

- Let $\boldsymbol{I}$ be a nonempty subset of S

I is a *left ideal* of S: $\boldsymbol{SI} \subseteq \boldsymbol{I}$.
I is a *right ideal* of S: $\boldsymbol{IS} \subseteq \boldsymbol{I}$.

I is a (*two-sided*) *ideal* of S: $\boldsymbol{SI} \cup \boldsymbol{IS} \subseteq \boldsymbol{I}$

- *Principal ideals* generated by an element a:

- *left*: $\mathbf{L}(\boldsymbol{a}) = S^1 a = a \cup Sa$:
- *right*: $\boldsymbol{R}(\boldsymbol{a}) = aS^1 = aS \cup a$
- *two-sided*: $\boldsymbol{J}(\boldsymbol{a}) = S^1 a S^1 = a \cup aS \cup Sa \cup SaS$

- Principal ideals generated by an idempotent e: $L(e) = Se$, $R(e) = eS$, $J(e) = SeS$.

Theorem 1. *The following conditions on a semigroup S, $E(S) \neq \varnothing$, are equivalent:*

(i) $(\forall e \in E(S))\ reg(Se) = Gr(Se)$;

(ii) $(\forall e \in E(S))\ reg(Se) = Reg(Se)$;

(iii) $Reg(S) = Gr(S)$.

M. Mitrović, *Regular Subsets of Semigroups Related to their Idempotents*, Semigroup Forum, Volume 70, Number 3 (2005), 356-360.

2. Decomposition of a semigroup

- Green's relations
- Simple semigroups
- Archimedean semigroups
- Hereditary archimedean semigroups
- Semilattice decompositions of semigroups
- Semilattices of archimedean semigroups - Putcha's semigroups
- *MBC*-semigroups

We want to divide the semigroup into subsets/subsemigroups in such a way that we can understand the semigroup in terms of those parts and their interaction.

• **Green's relations**, the fundamental tools in the structure theory of semigroups, **do not have nontrivial analogues** in groups, rings, quasigroups, lattices, universal algebras, fields.

• Relate elements depending on the ideals they generate, give a lot of information about the structure of a semigroup and how its elements interact.

$$
\begin{aligned}
a\,\mathcal{J}\,b &\Leftrightarrow J(a) = J(b)\\
a\,\mathcal{L}\,b &\Leftrightarrow L(a) = L(b)\\
a\,\mathcal{R}\,b &\Leftrightarrow R(a) = R(b)\\
\mathcal{H} &= \mathcal{L} \cap \mathcal{R}\\
\mathcal{D} &= \mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L}
\end{aligned}
$$

2.2 Green's equivalences

$$\triangle_S \subseteq \mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$$

Group: $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J} = \omega_G$.

Commutative semigroup: $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$.

$\mathcal{T}$-trivial: $\mathcal{T} = \triangle_S$, $\mathcal{T}$ is one of Green's relation.

Combinatorial, (group-free, torsion-free): $\mathcal{H} = \triangle_S$.

- **Green's subset(s)** of S:

$\boldsymbol{\mathcal{U}_{\mathcal{T}}(S)}$ - union of all $\mathcal{T}$-classes of S which are subsemigroups

M. Mitrović, **Semilattices of Archimedean Semigroups**, University of Niš - Faculty of Mechanical Engineering, Niš, 2003.

Various conditions on a semigroup can be expressed in terms of ideals, divisibility and Green's relations.

- $\mathcal{T} = S \times S = \omega_S$, $\mathcal{T}$ is one of Green's relations
  - S is *simple*: $\mathcal{J} = \omega_S \;\Leftrightarrow\; (\forall a, b \in S)\, a \in SbS$.
    - S is a *group*: $\mathcal{H} = \omega_S$ (unipotent simple)

- *Archimedean semigroups*: any two elements one of them divides some power of the other, i.e.
  $(\forall a, b \in S)(\exists n \in \mathbb{N})\ a^n \in J(b)$.

Lemma 2. *The S is an archimedean semigroup if and only if every left ideal of S is an archimedean semigroup.*

- T. Tamura (1975): The class of archimedean and the class of semilattices of archimedean semigroups **are not** subsemigroup closed

T. Tamura: *Quasi-orders, generalized Archimedeaness, semilattice decompositions*, Math. Nachr. 68, 1975, 201-220.

- S is a <u>*hereditary*</u> $\mathcal{K}$-<u>*semigroup*</u> if each its subsemigroup belongs to $\mathcal{K}$ or has a property $\mathcal{K}$

- S is ***hereditary archimedean***:

$$(\forall a, b \in S)(\exists n \in \mathbb{N})\ a^n \in \langle a, b\rangle b \langle a, b\rangle.$$

S. Bogdanović, M. Ćirić, M. Mitrović: *Semilattices of hereditary Archimedean semigroups*, 9:3, 1995, 611-617, in International Conference on Algebra, Logic & Discrete Math. Niš, April 14-16, 1995, ed. S. Bogdanović, M. Ćirić and Ž. Perović, Filomat.

Lemma 3. *A semigroup S is hereditary archimedean if and only if each subsemigroup of S is an archimedean semigroup.*

- $\mathfrak{N}$ is a *semilattice congruence* on a semigroup S if

  $Y = S/\mathfrak{N}$ is a semilattice
  $\mathfrak{N}$-class, S_α, $\alpha \in Y$, is a subsemigroup of S.

S is *a semilattice* Y *of semigroups* S_α, $\alpha \in Y$.

- Introduced by A. H. Clifford (1941).

- Special contributions: T. Tamura, N. Kimura and J. Shafer.
  · The series of their papers began in 1954
  · T. Tamura (1956, 1964):

Theorem 4. *Any semigroup is a semilattice of semilattice-indecomposable semigroups.*

- $\mathfrak{N}$, the smallest semilattice congruence on S, is connected to celebrated Green's relations in the following way:

$$\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J} \subseteq \mathfrak{N} \subseteq \omega_S.$$

The class of semilattice indecomposable semigroups even in the case of unipotent ones is somewhat complicated.

Proposition 5. *T. Tamura (1972) An archimedean semigroup is a semilattice-indecomposable.*

To the date the archimedean semigroups are the best known and the most popular class of semilattice-indecomposable semigroups.

To decompose given semigroup into semilattice of archimedean semigroups is a field of intensive research.

- T. Tamura (1954): any commutative semigroup is a semilattice of archimedean semigroups (commutativity means that components are t-archimedean).
- Semilattice decomposition provide the earliest structural insight into commutative semigroups in general.
- It has been the mainstay of commutative semigroup theory for many years.

P. A. Grillet: Commutative semigroups, Advances in Mathematics, Kluwer Academic Publishers, 2001.
A. Nagy: Special Classes of Semigroups, Springer-Science+Business Media, B. V., 2001. (Chapter 3)

- Semilattice decomposition of commutative semigroups into archimedean components has been applied usefully in study of commutative semigroup rings.

R. Gilmer, Commutative semigroup rings, The University of Chicago Press, 1984.

- *Putcha's semigroup* - a semigroup which is a semilattice of archimedean semigroups.

The first complete description is given in

M. S. Putcha, *Semilattice decompositions of semigroups*, Semigroup Forum 6 (1973), 12-34.

Theorem 6. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of archimedean semigroups;*

(ii) *for every $a, b \in S$, the assumption $a \mid b$ implies $a^2 \mid b^n$ for some $n \in \mathbb{N}$;*

(iii) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in Sa^2S\ ((ab)^n \in Sb^2S)$;

(iv) *$\sqrt{I}$ is an ideal of S for any ideal I of S.*

2.9 Mitrović-Bogdanović-Ćirić semigroups

- *MBC-semigroup* is a hereditary semilattice of archimedean semigroups.

Theorem 7. *A semigroup S is an MBC-semigroup if and only if*

$$(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in \langle a, b\rangle a^2 \langle a, b\rangle.$$

S. Bogdanović, M. Ćirić, M. Mitrović: *Semilattices of hereditary Archimedean semigroups*, 9:3, 1995, 611-617, in International Conference on Algebra, Logic & Discrete Math. Niš, April 14-16, 1995, ed. S. Bogdanović, M. Ćirić and Ž. Perović, Filomat.

- Description of some classes of semigroups which are subclass of the class of *MBC*-semigroups can be found in

M. Mitrović: Semilattices of Archimedean Semigroups, University of Niš, 2003.

- Particular examples of *MBC*-semigroups are described, for example, in

A. Nagy, *Special classes of semigroups*, Springer-Science+Business Media, B. V., 2001.

3. Decomposition of completely π-regular semigroups

- Archimedean components
- GVS-semigroups

Semigroups from this class which can be decomposed into archimedean components can be characterized from various points of view. Having in mind that the definition of finiteness condition may be given, also, in terms of elements of the semigroup, its subsemigroups, in terms of ideals or congruences of certain types, we choose to characterize them mostly by making connections between their elements and/or their special subsets.

3.1 The most popular semigroups with finiteness conditions

S is $\underline{\textit{band}}$: $\boldsymbol{S = E(S)}$

S is $\underline{\textit{regular}}$: $\boldsymbol{S} = \boldsymbol{Reg(S)}$

S is $\underline{\textit{completely regular}}$: $\boldsymbol{S} = \boldsymbol{Gr(S)}$

S is $\boldsymbol{\pi}$-$\underline{\textit{regular}}$: $\boldsymbol{S = \sqrt{Reg(S)}}$

S is $\underline{\textit{completely}}$ $\boldsymbol{\pi}$-$\underline{\textit{regular}}$: $\boldsymbol{S = \sqrt{Gr(S)}}$

S is $\underline{\textit{periodic}}$: $\boldsymbol{S} = \boldsymbol{\sqrt{E(S)}}$

Theorem 8. *S is completely π-regular if and only if for any $a \in S$ there exists $n \in \mathbb{N}$ such that $a^n \in a^n S a^{n+1}$ ($a^n \in a^{n+1} S a^n$).*

Class $\mathcal{C}\pi\mathcal{R}$ is very large. It includes:

- class of bands, $\mathcal{B}$;
- class of completely regular semigroups, $\mathcal{CR}$;
- class of finite semigroups, $\mathcal{FS}$;
- class of periodic semigroups, $\mathcal{P}$.

It also contains some important concrete semigroups, like:

- the semigroup of all matrices over a division ring.

Completely π-regular semigroups naturally appear in ring theory.

The multiplicative semigroup of every semisimple Artinian ring is a completely π-regular semigroup. Since semisimple Artinian rings and their generalizations play key roles in many ring theorems, some facts concerning the structure of completely π-regular semigroups become useful in deducing properties of rings.

Rings graded by completely π-regular semigroups of various types has been investigated by many authors too.

J. Okniński, *Smigroup algebras*, Marcel Dekker, New York, 1991.

A. V. Kelarev, *Applications of epigroups to graded ring theory*, Semigroup Forum 50 (1995), 327-350.

- ***Galbiati-Veronesi-Shevrin semigroup***, shortly *GV S*-semigroup: a semilattice of completely archimedean semigroups.

Studies of decompositions of completely π-regular semigroups into semilattices of archimedean semigroups began in L. N. Shevrin's papers. The final results of his several year long investigations are given in [1].

Similar results concerning decompositions of completely π-regular semigroups into semilattices of archimedean semigroups were obtained by J. L. Galbiati and M. L. Veronesi, [2], where they started such investigations, and by M. L. Veronesi in paper [3], where it ended.

[1] L. N. Shevrin: *Theory of epigroups I*, Mat. Sb. 185 no 8 (1994), 129-160, (in Russian).

[2] J. L. Galbiati, M. L. Veronesi: *Semigruppi quasi regolari*, Atti del convegno: Teoria dei semigruppi, Siena, 1982, 91-95, (Ed. F. Migliorini).

[3] M. L. Veronesi: *Sui semigruppi quasi fortemente regolari*, Riv. Mat. Univ. Parma, (4) 10, (1984), 319-329.

3.5 *GV S*-semigroups

Theorem 11. *The following conditions on a semigroup S are equivalent:*

(i) *S is a GVS-semigroup;*

(ii) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in (ab)^n bS(ab)^n$;

(iii) *S is π-regular and $Reg(S) = Gr(S)$;*

(iv) *S is completely π-regular and $Gr(S) = \mathcal{U}_{\mathcal{J}}(S) = \mathcal{U}_{\mathcal{L}}(S)$.*

- *The nilpotent part* of S - the set of all nilpotent elements

$$Nil(S) = \sqrt{\{0\}} = \{x \in S : (\exists n \in \mathbb{N})\, x^n = 0\}$$

Lemma 10. *If $S = S^0$ is a semilattice of archimedean semigroups then $Nil(S)$ is an ideal of S.*

- $(R, +, \cdot)$ is *strongly π-regular* if

  $R_\bullet = (R, \cdot)$ is completely π-regular semigroup

Theorem 11. *The following conditions on a strongly π-regular ring R are equivalent:*

(i) $R_\bullet$ *is a GVS-semigroup;*

(ii) *Nil(R) is an ideal of* $R_\bullet$*;*

(iii) *Nil(R) is an (ring) ideal of R.*

Example 12. Ring of all $n \times n$ triangular matrices over field F, $n \in \mathbb{N}$

M. S. Putcha, *Rings which are semilattices of archimedean semigroups*, Semigroup Forum 23, 1981, 1-5

4. Decomposition of periodic semigroups

- Hereditary *GVS*-semigroups
- Combinatorial periodic semigroups
- Combinatorial *GVS*-semigroups

Semigroups from this class which can be decomposed into archimedean components can be characterized from various points of view. Having in mind that the definition of finiteness condition may be given, also, in terms of elements of the semigroup, its subsemigroups, in terms of ideals or congruences of certain types, we choose to characterize them mostly by making connections between their elements and/or their special subsets.

4.1 Periodic semigroups

- S is ***periodic***: $S = \sqrt{E(S)}$

  Class $\mathcal{P}$ all periodic semigroups includes:

  - class of bands, $\mathcal{B}$;
  - class of finite semigroups, $\mathcal{FS}$;
  - class of nil semigroups, $\mathcal{N}$.

Theorem 13. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of completely hereditary archimedean semigroups;*

(ii) *S is a semilattice of nil-extensions of periodic completely simple semigroups;*

(iii) $(\forall a, b \in S)(\exists k \in \mathbb{N})\ (ab)^k = (ab)^k((ba)^k(ab)^k)^k$;

(iv) *S is periodic and $\boldsymbol{Reg(S) = Gr(S)}$;*

(v) *S is hereditary* GVS-semigroup (every subsemigroup of S is GVS-semigroup);

(vi) $\boldsymbol{reg(T) = Reg(T) \neq \varnothing}$, for any subsemigroup T of S.

Hereditary GVS-semigroups and periodic MBC-semigroups coincide.

M. Mitrović: Semilattices of Archimedean Semigroups, University of Niš, 2003.

(*Torsion-free* or *group-free*) semigroups

- In terms of equality of regular subsets: $E(S) = Gr(S)$.
- In terms of Green's relations: $\mathcal{H}$-*trivial* semigroup, i.e. $\mathcal{H} = \triangle_S$.

4.4 Combinatorial periodic semigroups

The class of all combinatorial periodic semigroups includes the class $\mathcal{B}$ of all bands.

Within semigroup graded ring theory combinatorial periodic semigroups are often called *power-stationary semigroups*.

Prime radicals and radical of rings graded by such type of semigroups are considered in

A. D. Bell, S. S. Stadler, M. L. Teply, *Prime ideals and radicals in semigroup-graded rings*, Proc. Edinburgh Math. Soc. 39, 1996, 125.

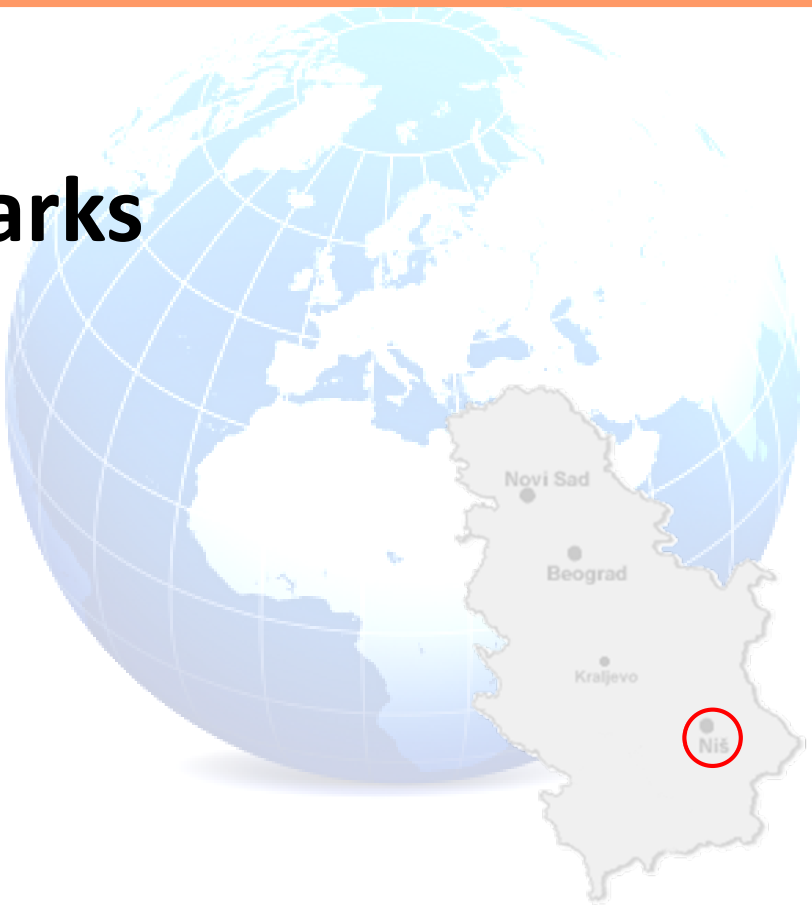
Theorem 15. *The following conditions on a semigroup S are equivalent:*

(i) *S is a combinatorial GVS-semigroup;*

(ii) *S is $\boldsymbol{\pi}$-regular and $\boldsymbol{E(S) = Reg(S)}$;*

(iii) $(\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^{2n+1} = (ab)^n ba^2 (ab)^n$.

5. Concluding remarks



Semilattice-graded ring:

$$R = \oplus_{\alpha \in Y} R[S_\alpha],$$

S is a semilattice Y of semigroups S_α, $\alpha \in Y$,

$R[S_\alpha]$ has simpler structure.

- Such type of rings allow us to carry over the information from $R[S_\alpha]$ to $R[S]$.

- This method, used by many authors, allows possibility a semigroup S belongs to one of the class of semilattices of archimedean semigroups just described.

THANK YOU FOR YOUR ATTENTION!