# Graded von Neumann regular rings

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# Overview

1. Graded von Neumann regular rings (Năstăsescu, Oystaeyen, 1982)
2. nearly epsilon-strongly graded rings (Nystedt, Öinert, 2018)

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2. nearly epsilon-strongly graded rings (Nystedt, Öinert, 2018)

$$\{\text{Graded von Neumann regular rings}\} \subsetneq \{\text{nearly epsilon-strongly graded rings}\}$$

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# Reference

1. Daniel Lännström, *A characterization of graded von Neumann regular rings with applications to Leavitt path algebras*, In preparation.

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# Von Neumann regular rings

**Definition (Von Neumann, 1936)**

A unital ring $R$ is called *von Neumann regular* if for each $a \in R$ there is some $x \in R$ such that $a = axa$.

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## Proposition

Let $R$ be an s-unital ring. The following are equivalent:

1. $R$ is *von Neumann regular*
2. every finitely generated left (right) ideal is generated by an idempotent

# Group graded rings

## Definition

Let $G$ be a group and let $S$ be a ring. A *grading* of $S$ is a collection of additive subsets of $S$, $\{S_g\}_{g\in G}$, such that

$$S = \bigoplus_{g\in G} S_g,$$

and $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$. The ring $S$ is called a *$G$-graded ring*.

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## Definition

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2. The elements $\bigcup_{g \in G} S_g$ are called the *homogeneous elements* of $S$.
3. An ideal $I$ of $S$ is called *graded* if $I = \bigoplus_{g \in G}(I \cap S_g)$.

# Graded rings: Examples I

## Example

The Laurent polynomial ring is $\mathbb{Z}$-graded by,

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## Example

(The group ring) Let $G$ be a group and let $R$ be a unital ring. The group ring $R[G] = \bigoplus_{g \in G} R\delta_g$ is naturally $G$-graded.
Principal component: $R$

# Graded von Neumann regular rings

**Definition (Năstăsescu, Oystaeyen, 1982 [2])**

Let $S = \bigoplus_{g \in G} S_g$ be a $G$-graded ring. If, for each $g \in G$ and every $a \in S_g$, there is some $x \in S$ such that $a = axa$, then $S$ is called *graded von Neumann regular*.

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## Proposition (Hazrat, 2014)

Let $S$ be a $G$-graded ring with homogeneous local units. Then the following are equivalent:

1. $S$ is graded von Neumann regular
2. Any finitely generated right (left) graded ideal of $S$ is generated by a homogeneous idempotent.

# Graded von Neumann regular rings and strongly graded rings

## Definition

A $G$-grading $S = \bigoplus_{g \in G} S_g$ is called *strong* if $S_g S_h = S_{gh}$ for all $g, h \in G$.

## Theorem (Năstăsescu, Oystaeyen, 1982 [2])

Let $S = \bigoplus_{g \in G} S_g$ be a unital strongly $G$-graded ring. Then $S$ is graded von Neumann regular if and only if $S_e$ is von Neumann regular.

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**Example**

Let $R$ be a von Neumann regular ring. Then the group ring $R[G]$ and the Laurent polynomial ring $R[x, x^{-1}]$ are graded von Neumann regular.

# Epsilon-strongly graded rings

Examples of epsilon-strongly graded rings:

1. unital partial crossed products (Nystedt, Öinert, Pinedo, 2016 [3]),
2. Leavitt path algebras of finite graphs (Nystedt, Öinert, 2017),
3. corner skew Laurent polynomial rings (Lännström, 2019).

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## Definition (Nystedt, Öinert, 2018)

Let $S$ be a $G$-graded ring. If, for every $g \in G$ and $s \in S_g$ there exist some $\epsilon_g(s) \in S_g S_{g^{-1}}, \epsilon_g(s)' \in S_{g^{-1}} S_g$ such that $\epsilon_g(s)s = s = s\epsilon_g(s)'$, then $S$ is called *nearly epsilon-strongly $G$-graded*.

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# Necessary conditions for being graded von Neumann regular

## Proposition

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Let $S = \bigoplus_{g \in G} S_g$ be a graded von Neumann regular ring. Then $S$ is nearly epsilon-strongly $G$-graded.

## Proof

Take $g \in G$ and $s \in S_g$. We need to find some $\epsilon_g(s) \in S_g S_{g^{-1}}$ and $\epsilon_g(s)' \in S_{g^{-1}} S_g$ such that $\epsilon_g(s) s = s = \epsilon_g(s)' s$.

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Take $g \in G$ and $s \in S_g$. We need to find some $\epsilon_g(s) \in S_g S_{g^{-1}}$ and $\epsilon_g(s)' \in S_{g^{-1}} S_g$ such that $\epsilon_g(s)s = s = \epsilon_g(s)'s$.
Since $S$ is graded von Neumann regular, there is some $b \in S_{g^{-1}}$ such that $s = sbs$.
Take $\epsilon_g(s) := sb$ and $\epsilon_g(s)' = bs$.

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Necessary conditions are also sufficient! (cf. Yahya, 1997)

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**Theorem (Lännström, 2019)**

*Let $S = \bigoplus_{g \in G} S_g$ be a $G$-graded ring. Then $S$ is graded von Neumann regular if and only if $S$ is nearly epsilon-strongly $G$-graded and $S_e$ is von Neumann regular.*

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**Remark**

Generalizes Năstăsescu, Oystaeyen theorem for unital strongly graded rings!

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Generalizes Năstăsescu, Oystaeyen theorem for unital strongly graded rings!

## Acknowledgement

I am grateful to Öinert for providing part of the proof.

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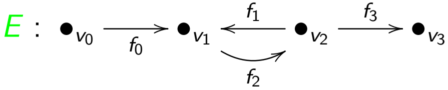

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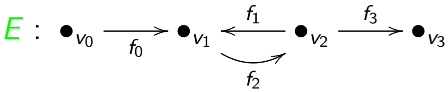


The Leavitt path algebra $L_R(E)$ is a $\mathbb{Z}$-graded $R$-algebra.

# Research question

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## Research question

How does algebraic properties of $R$ effect the algebraic properties of $L_R(E)$?

# Leavitt path algebras: Examples I

## Example

$$A_1 : \qquad \bullet_v$$

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## Example

$E_1 :$


In this case, $L_R(E_1) \cong_\phi R[x, x^{-1}]$ via the map defined by $\phi(v) = 1_R, \phi(f) = x, \phi(f^*) = x^{-1}$.

# Leavitt path algebras: Examples II

The previous graphs have all been finite, but we also allow infinite graphs!

**Example**

Infinitely many vertices:

$$E' : \bullet_{v_1} \quad \bullet_{v_2} \quad \bullet_{v_3} \quad \bullet_{v_4} \quad \bullet_{v_5} \quad \bullet_{v_6} \quad \bullet_{v_7} \quad \bullet_{v_8} \quad \bullet_{v_9} \quad \bullet_{v_{10}} \cdots$$

In this case, $L_R(E') \cong \bigoplus_{i>0} Rv_i$.

**Example**

$$E'' : \bullet_{v_1} \xrightarrow{(\infty)} \bullet_{v_2}$$

# Leavitt path algebras over fields are graded von Neumann regular

**Theorem (Hazrat, 2014 [1])**

Let $K$ be a field and let $E$ be a directed graph. Then $L_K(E)$ is graded von Neumann regular.

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## Theorem (Hazrat, 2014 [1])

Let $K$ be a field and let $E$ be a directed graph. Then $L_K(E)$ is graded von Neumann regular.

## Example

Let $R$ be a unital ring that is not von Neumann regular.

$$A_1 : \qquad \bullet_v$$

It can be shown that $L_R(E) \cong_{\mathrm{gr}} R$ is graded von Neumann regular if and only if $R$ is von Neumann regular. Hence, $L_R(E)$ is not graded von Neumann regular.

# Leavitt path algebras with coefficients in a general unital ring

Main goal of this project:

**Theorem (Lännström, 2019)**

Let $R$ be a unital ring and let $E$ be a directed graph. Then $L_R(E)$ is graded von Neumann regular if and only if $R$ is von Neumann regular.

# Leavitt path algebras with coefficients in a general unital ring

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Let $R$ be a unital ring and let $E$ be a directed graph. Then $L_R(E)$ is graded von Neumann regular if and only if $R$ is von Neumann regular.

Proof idea:
$R$ von Neumann regular $\rightsquigarrow$ $L_R(E)_0$ is von Neumann regular $\rightsquigarrow$ $L_R(E)$ graded von Neumann regular.

# Leavitt path algebras with coefficients in a general unital ring

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$R$ von Neumann regular $\rightsquigarrow$ $L_R(E)_0$ is von Neumann regular $\rightsquigarrow$ $L_R(E)$ graded von Neumann regular.

**Proposition (Nystedt, Öinert, 2018)**

Let $R$ be a unital ring. If $E$ is a directed graph, then $L_R(E)$ is nearly epsilon-strongly $\mathbb{Z}$-graded. If $E$ is a finite directed graph, then $L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded.

# Example

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By Theorem, for any graph $E$,

1. $L_{\mathbb{C}}(E)$ is graded von Neumann regular
2. $L_{\mathbb{Z}}(E)$ is not graded von Neumann regular

# Proof of Theorem (I)

## Lemma (Lännström, 2019)

Let $R$ be a unital ring and let $E$ be a finite directed graph. Then $(L_R(E))_0$ is von Neumann regular if and only if $R$ is von Neumann regular.

# Proof of Theorem (I)

## Lemma (Lännström, 2019)

Let $R$ be a unital ring and let $E$ be a finite directed graph. Then $(L_R(E))_0$ is von Neumann regular if and only if $R$ is von Neumann regular.

## Corollary (Lännström, 2019)

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## Proof.

$L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded (Nystedt, Öinert, 2017).

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## Proof.

$L_R(E)$ is epsilon-strongly $\mathbb{Z}$-graded (Nystedt, Öinert, 2017).
$L_R(E)$ is graded von Neumann regular if and only if $(L_R(E))_0$ is von Neumann regular.
By Lemma, $(L_R(E))_0$ is von Neumann regular if and only if $R$ is von Neumann regular. □

# Proof of Theorem (II)

Technical reduction step: Let $E$ be any graph. Then $L_R(E)$ is the direct limit of Leavitt path algebras of finite graphs.

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**Theorem (Lännström, 2019)**

Let $R$ be a unital ring and let $E$ be any (possibly non-finite) directed graph. Then $L_R(E)$ is graded von Neumann regular if and only if $R$ is von Neumann regular.

# Semiprimitive and semiprime Leavitt path algebras

## Corollary (Lännström, 2019)

*Let $R$ be a unital ring and let $E$ be a directed graph. If $R$ is von Neumann regular, then $L_R(E)$ is semiprimitive and semiprime.*

## Proof.

Assume that $R$ is von Neumann regular. By Theorem, $L_R(E)$ is graded von Neumann regular.

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Assume that $R$ is von Neumann regular. By Theorem, $L_R(E)$ is graded von Neumann regular.
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Assume that $R$ is von Neumann regular. By Theorem, $L_R(E)$ is graded von Neumann regular.
Since $L_R(E)$ is $\mathbb{Z}$-graded with local units, $J(L_R(E))$ is a graded idea. (Bergman's theorem).
Using the properties of graded von Neumann regular rings, $J = 0$ and, moreover, $L_R(E)$ is semiprime. □

# References I


R. Hazrat. "Leavitt path algebras are graded von Neumann regular rings". In: *Journal of Algebra* 401 (2014), pp. 220–233.

C. Năstăsescu and F. van Oystaeyen. "Graded ring theory, Math". In: *Library (28), North-Holland, Amsterdam* (1982).

P. Nystedt, J. Öinert, and H. Pinedo. "Epsilon-strongly graded rings, separability and semisimplicity". In: *Journal of Algebra* 514 (2018), pp. 1–24. ISSN: 0021-8693. DOI: https://doi.org/10.1016/j.jalgebra.2018.08.002.

*Thank you for your attention!*