On K-invariant q-deformed Levi-Civita connection

Kwalombota Ilwale

Department of Mathematics(MAI), Linköping University

October 18, 2019

My Advisors

Joakim Arnlind
Main advisor
Linkoping University

Vladimir Tkatjev
Assistant advisor
Linkoping University

Outline

Outline

Outline

Outline

Outline

Outline

The quantum 3-sphere

Let S_q^3 denote the unital $*$-algebra generated by a, a^*, c, c^* fulfilling

$$ac = qca \qquad c^*a^* = qa^*c^* \qquad ac^* = qc^*a$$
$$ca^* = qa^*c \qquad cc^* = c^*c \qquad a^*a + c^*c = aa^* + q^2cc^* = 1.$$

The Hopf algebra structure is given by

$$\Delta(a) = a \otimes a - qc^* \otimes c \qquad \Delta(c) = c \otimes a + a^* \otimes c$$
$$\Delta(a^*) = -qc \otimes c^* + a^* \otimes a^* \qquad \Delta(c^*) = a \otimes c^* + c^* \otimes a^*$$

with antipode and counit

$$S(a) = a^* \qquad S(c) = -qc \qquad S(a^*) = a \qquad S(c^*) = -q^{-1}c^*$$
$$\epsilon(a) = 1 \qquad \epsilon(c) = 0 \qquad \epsilon(a^*) = 1 \qquad \epsilon(c^*) = 0.$$

Outline

The Quantum algebra $U_q(su(2))$

$U_q(su(2))$ is a $*$-algebra generated by E, F, K, K^{-1} satisfying

$$K^{\pm}E = q^{\pm 1}EK \qquad [E,F] = \frac{K^2 - K^{-2}}{q - q^{-1}}$$

$$K^{\pm}F = q^{\mp 1}FK$$

$$(K^{\pm 1})^* = K^{\pm 1} \qquad E^* = F \qquad F^* = E.$$

The Hopf algebra structure is given by

$$\Delta(E) = E \otimes K + K^{-1} \otimes E \qquad \Delta(F) = F \otimes K + K^{-1} \otimes F$$

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}$$

There is a unique bilinear pairing between $U_q(su(2))$ and S_q^3 inducing $U_q(su(2))$-bimodule structure on S_q^3.

Outline

The Quantum tangent space TS_q^3

The quantum tangent space TS_q^3 is a vector space spanned by

$$X_+ = \sqrt{q}EK, \qquad X_- = \frac{1}{\sqrt{q}}FK, \qquad X_z = \frac{1-K^4}{1-q^{-2}}$$

satisfying a q-deformed Leibniz's rule

$$\begin{aligned} X_+(fg) &= fX_+(g) + X_+(f)K^2(g) \\ X_-(fg) &= fX_-(g) + X_-(f)K^2(g) \\ X_z(fg) &= fX_z(g) + X_z(f)K^4(g) \end{aligned}$$

for $f, g \in S_q^3$.

Motivation

Let M be a free S_q^3-module with basis $e_1, e_2, \ldots, e_n$. We set

$$\nabla_X^0 m = X(m^a)e_a \quad \text{and} \quad K(m) = K(m^a)e_a$$

for $m = m^a e_a \in M, X \in TS_q^3$ and $K \in U_q(su(2))$. Immediately we have

$$\begin{aligned} \nabla_+^0(fm) &= f\nabla_+^0 m + X_+(f)K^2(m) \\ \nabla_-^0(fm) &= f\nabla_-^0 m + X_-(f)K^2(m) \\ \nabla_z^0(fm) &= f\nabla_z^0 m + X_+(f)K^4(m) \end{aligned}$$

For example,

$$\begin{aligned} \nabla_+^0(fm) &= X_+(fm^a)e_a \\ &= fX_+(m^a)e_a + X_+(f)K^2(m^a)e_a \\ &= f\nabla_+^0 m + X_+(f)K^2(m). \end{aligned}$$

Definitions

Definition

Let M be a left S_q^3-module and let $K : M \longrightarrow M$ be a map such that

$$K(m_1 + m_2) = K(m_1 + m_2)$$
$$K(fm) = K(f)K(m)$$

for $f \in S_q^3$ and $m, m_1, m_2 \in M$. Then M is called a module with a K-action.

Definitions

Definition

Let M be a left S_q^3- module with a K-action. A left q-affine connection on M is a map $\nabla : TS_q^3 \times M \longrightarrow M$ such that

$$\nabla_X(m_1 + m_2) = \nabla_X m_1 + \nabla_X m_2$$
$$\nabla_{\lambda X + Y} m = \lambda \nabla_X m + \nabla_Y m$$
$$\nabla_\pm(fm) = f\nabla_\pm m + X_\pm(f)K^2(m)$$
$$\nabla_z(fm) = f\nabla_z m + X_z(f)K^4(m)$$

for $m, m_1, m_2 \in M, f \in S_q^3, X \in TS_q^3$ and $\lambda \in \mathbb{C}$.

Definitions

q-affine connections exists on projective modules.
We assume that $[K,P](m)=0$ for all $m\in(S_q^3)^n$ where
$P:(S_q^3)^n\longrightarrow(S_q^3)^n$ is a projector.
Then $M=P((S_q^3)^n)$ is a projective module with a K-action, and

$$\nabla_X m = P(\nabla_X^0 m)$$

is a q-affine connection on M. For example,

$$\begin{aligned}\nabla_\pm(fm)=P(\nabla_\pm^0(fm)) &= fP(\nabla_\pm^0 m)+X_\pm(f)PK^2(m))\\ &= fP(\nabla_\pm^0 m)+X_\pm(f)K^2P(m))\\ &= fP(\nabla_\pm^0 m)+X_\pm(f)K^2(m))\\ &= f\nabla_\pm m+X_\pm(f)K^2(m).\end{aligned}$$

Let us consider a hermitian form on the free S_q^3-module with $h(e_a, e_b) = \delta_{ab}$. It is reasonable to set

$$h(m_1, m_2) = m_1^a \delta_{ab} (m_2^b)^\star$$

for $m_1, m_2 \in M$ and $m_1^a, m_2^b \in S_q^3$. Furthermore

$$\begin{aligned} Kh(m_1, m_2) &= K(m_1^a \delta_{ab} (m_2^b)^\star) \\ &= K(m_1^a) K(\delta_{ab}) K((m_2^b)^\star) \\ &= K(m_1^a) \delta_{ab} K^{-1} (m_2^b)^\star \\ &= h(K(m_1), K^{-1}(m_2)). \end{aligned}$$

Definition

A hermitian form h on a module M with a K-action is called K-invariant if

$$K(h(m_1, m_2)) = h(K(m_1), K^{-1}(m_2)),$$

for $m_1, m_2 \in M$.

Motivated by the properties of ∇^0, we have the following definition.

Definition

A q-affine connection ∇ on M is compatible with the hermitian form $h : M \times M \longrightarrow S_q^3$ if

$$X_{\pm}(h(m_1, m_2)) = -h(m_1, K^{-2}(\nabla_{\mp} m_2)) + h(\nabla_{\pm} m_1, K^{-2}(m_2))$$
$$X_z(h(m_1, m_2)) = -h(m_1, K^{-4}(\nabla_z m_2)) + h(\nabla_z m_1, K^{-4}(m_2))$$

for $m_1, m_2 \in M$.

Definitions

Let the module of 1-forms $\Omega^1(S_q^3)$ be a free module over S_q^3 with basis given by

$$\omega_1 = \omega_+ \qquad \omega_2 = \omega_- \qquad \omega_3 = \omega_z$$

together with the differential $d : S_q^3 \longrightarrow \Omega^1(S_q^3)$

$$df = X_+(f)\omega_+ + X_-(f)\omega_- + X_z(f)\omega_z$$

for $f \in S_q^3$.
$\Omega^1(S_q^3)$ is a bimodule with respect to the relations

$$\omega_z a = q^{-2} a\omega_z \quad \omega_z a^\star = q^2 a^\star \omega_z \quad \omega_z c = q^{-2} c\omega_z \quad \omega_z c^\star = q^2 c^\star \omega_z$$
$$\omega_\pm a = q^{-1} a\omega_\pm \quad \omega_\pm a^\star = q^\star a\omega_\pm \quad \omega_\pm c = q^{-1} c\omega_\pm \quad \omega_\pm c^\star = qc^\star \omega_\pm .$$

Definitions

Lt us set $K(\omega_a) = l_a \omega_a$ and $K(f\omega_a) = K(f)K(\omega_a)$ where $l_1, l_2, l_3 \in \mathbb{R} - \{0\}$.
A hermitian form h on $\Omega^1(S_q^3)$ being determined by $h_{ab} = h(\omega_a, \omega_b)$ with $h(m_1, m_2) = m_1^a h_{ab} (m_2^b)^\star$ where $m_i = m_i^a \omega_a$ for $i = 1, 2$.

(1) If h is K-invariant then $K(h_{ab}) = l_a l_b^{-1} h_{ab}$.

(2) If we assume that the K-invariant hermitian form h is diagonal with $h_{aa} = h_a$, $K(h_a) = h_a$ and invertible in the sense that h_a^{-1} exist for all a, then

$$h_{++} = h_+, \qquad h_{--} = h_- \qquad h_{zz} = h_z$$
$$h_{ab} = 0 \quad \text{if} \quad a \neq b.$$

Definition

A q-affine connection on $\Omega^1(S_q^3)$ is torsion free if

$$\begin{aligned}
\nabla_-\omega_+ - q^2\nabla_+\omega_- &= \omega_z \\
q^2\nabla_z\omega_- - q^{-2}\nabla_-\omega_z &= (1+q^2)\omega_- \\
q^2\nabla_+\omega_z - q^{-2}\nabla_z\omega_+ &= (1+q^2)\omega_+.
\end{aligned}$$

This definition is motivated by the commutation relations

$$\begin{gathered}
X_-X_+ - q^2X_+X_- = X_z \\
q^2X_zX_- - q^{-2}X_-X_z = (1+q^2)X_- \\
q^2X_+X_z - q^{-2}X_zX_+ = (1+q^2)X_+.
\end{gathered}$$

Since $\Omega^1(S_q^3)$ is a free module, a q-affine connection on $\Omega^1(S_q^3)$ is determined by specifying the Christoffel symbols $\Gamma^a_{bc} \in S_q^3$ given by $\nabla_a \omega_b = \Gamma^c_{ab}\omega_c$. For example, when $a = + = b$ one obtains one of the metric compatible equations

$$X_+ h_+ \;=\; -l_+^2 h_+ K^{-2} (\Gamma^+_{-+})^\star + l_+^2 \Gamma^+_{++} h_+,$$

and from torsion free equation, one obtains

$$\begin{aligned} \nabla_- \omega_+ - q^2 \nabla_+ \omega_- &= \omega_z \\ \Gamma^+_{-+} - q^2 \Gamma^+_{+-} &= 0, \\ \Gamma^-_{-+} - q^2 \Gamma^-_{+-} &= 0, \\ \Gamma^z_{-+} - q^2 \Gamma^z_{+-} &= 1 \end{aligned}$$

There are 15 metric equations and 9 torsion free equations involving 27 Christofel symbols.

There is a Levi-Civita connection on $\Omega^1(S^3_q)$ given by the follwing equations:

$$\begin{aligned}
\Gamma^{+}_{-+} &= q^4 l_+^2 l_-^{-2} \Gamma^{-}_{--} h_- h_+^{-1} - q^4 l_+^2 X_-(h_-) h_+^{-1} \\
\Gamma^{+}_{+-} &= q^2 l_+^2 l_-^{-2} \Gamma^{-}_{--} h_- h_+^{-1} - q^2 l_+^2 X_-(h_-) h_+^{-1} \\
\Gamma^{-}_{-+} &= q^2 h_- K^{-2} (\Gamma^{-}_{--})^\star h_-^{-1} + q^2 l_-^2 X_+(h_-) h_-^{-1} \\
\Gamma^{-}_{+-} &= h_- K^{-2} (\Gamma^{-}_{--})^\star h_-^{-1} + l_-^2 X_+(h_-) h_-^{-1} \\
\Gamma^{+}_{++} &= q^4 l_+^2 l_-^{-2} h_- K^{-2} (\Gamma^{-}_{--})^\star h_+^{-1} + l_+^2 X_+(q^4 h_- + h_+) h_+^{-1} \\
\Gamma^{-}_{++} &= l_-^{-2} l_+^{-2} h_+ K^{-2} (\Gamma^{+}_{--})^\star h_-^{-1} \\
\Gamma^{z}_{+-} &= l_z^2 l_-^{-2} K^{-2} (\Gamma^{-}_{-z}) h_- h_z^{-1} \\
\Gamma^{z}_{-+} &= \mathbf{1} + q^2 l_z^2 l_-^{-2} K^{-2} (\Gamma^{-}_{-z}) h_- h_z^{-1} \\
\Gamma^{+}_{+z} &= q^{-2} l_+^2 l_-^{-2} \Gamma^{-}_{-z} h_- h_+^{-1} - q^{-2} l_+^2 l_z^{-2} h_z h_+^{-1} \\
\Gamma^{+}_{z+} &= q^2 l_+^2 l_-^{-2} \Gamma^{-}_{-z} h_- h_+^{-1} - q^2 l_+^2 l_z^{-2} h_z h_+^{-1} - q^2(1+q^2)\mathbf{1} \\
\Gamma^{-}_{z-} &= q^{-4} h_- K^{-4} (\Gamma^{-}_{-z})^\star h_-^{-1} + q^{-2}(1+q^2)\mathbf{1}
\end{aligned}$$

$$
\begin{aligned}
\Gamma^{z}_{zz} &= h_z K^{-4}(\tilde{\Gamma}^{z}_{zz})^{\star} h_z^{-1} \\
\Gamma^{z}_{-z} &= q^{-4} K^{-2}(\Gamma^{+}_{zz}) h_+ h_z^{-1} + X_-(h_z) h_z^{-1} \\
\Gamma^{z}_{z-} &= q^{-8} K^{-2}(\Gamma^{+}_{zz}) h_+ h_z^{-1} + q^{-4} X_-(h_z) h_z^{-1} \\
\Gamma^{z}_{+z} &= q^{-4} h_+ K^{-4}(\Gamma^{+}_{zz})^{\star} h_z^{-1} \\
\Gamma^{z}_{z+} &= h_+ K^{-4}(\Gamma^{+}_{zz})^{\star} h_z^{-1} \\
\Gamma^{-}_{zz} &= q^{-8} h_+ K^{-6}(\Gamma^{+}_{zz})^{\star} h_-^{-1} - q^{-4} K^{2} X_+(h_z) h_-^{-1} \\
\Gamma^{+}_{-z} &= q^{8} K^{2}(\Gamma^{z}_{--}) h_z h_+^{-1} \\
\Gamma^{+}_{z-} &= q^{4} K^{2}(\Gamma^{z}_{--}) h_z h_+^{-1} \\
\Gamma^{-}_{+z} &= h_z K^{-2}(\Gamma^{z}_{--})^{\star} h_-^{-1} \\
\Gamma^{-}_{z+} &= q^{4} h_z K^{-2}(\Gamma^{z}_{--})^{\star} h_-^{-1} \\
\Gamma^{z}_{++} &= q^{8} h_z (\Gamma^{z}_{--})^{\star} h_z^{-1}
\end{aligned}
$$

Outlook

- Properties of curvature
- Are there conditions to obtain a unique Levi-Civita connection?
- This can be done for other quantum groups.
- Bimodule connection

- A. Klimyk and K Schudgen, Quantum groups, *Springer*, 1998.
- G. Landi, Twisted sigma - model solitons on the Quantum projective line, *Lett. Math. Phys*. **108**, (8) 1995 - 1983 (2018).
- P. Podles, Introduction to Quantum groups, *Rev. Math . Phys***10**, (3), 551 -551–876 (1998).
- S. L. Woronowicz, Twisted $SU(2)$ group. An example of a non-commutative differential calculus, *RIMS, Kyoto Univ.* **23**, (3), 117 - 181(1987).
- S. L. Woronowicz, Compact matrix Pseudogroup, *Commun. Math. Phys.* **111**, 613 - 665(1987).

Thank you for your Attention

Thank you!