# Commutative nonassociative algebras, representations of finite groups and minimal cones

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# Introduction

The most straightforward approach to an algebra (like $\mathbb{C}$ or $\mathbb{H}$) is to describe it by a multiplication table. Sometimes one define an algebra via an identity (like Lie and Jordan algebras). But, if you go beyond small dimensions, this approach is not very satisfactory.

Let $A$ be an commutative nonassociative algebra over $\mathbb{F}$ decomposed into a direct sum

$$A = A_1 \oplus A_2 \oplus \ldots A_m.$$

The most interesting case is when $A_i$ are invariant subspaces of the adjoint operator

$$\mathrm{ad}_c : x \to cx, \quad c \text{ being an idempotent in } A.$$

A *fusion law* or *fusion rule* is a map $\star : (i,j) \to i \star j$ such that

$$A_i A_j \subset \bigoplus_{k \in i \star j} A_k.$$

A fusion law is 'nice' if both $m$ and the cardinality of $i \star j$ are reasonably small. There are only few known examples with nice (i.e. graded) fusion laws. All of them have important features and connections to different areas of mathematics.

# A key example: Jordan algebra of symmetric matrices

A commutative algebra $A$ is **Jordan** if $x^2(xy) = x(x^2y)$ for all $x, y \in A$.

Let $A$ be the 3-dimensional Jordan algebra of real symmetric $2 \times 2$-matrices with multiplication

$$x \bullet y = \frac{1}{2}(xy + yx)$$

Note that $x \bullet x = xx = x^2$. Set $c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

| $\bullet$ | $e_1$ | $e_2$ | $e_3$ |
|---|---|---|---|
| $e_1$ | $e_1$ | $0$ | $\frac{1}{2}e_3$ |
| $e_2$ | $0$ | $e_2$ | $\frac{1}{2}e_3$ |
| $e_3$ | $\frac{1}{2}e_3$ | $\frac{1}{2}e_3$ | $e_1 + e_2$ |

$\Rightarrow \{e_1, e_2, e_3\}$ is an eigenbasis of $\mathrm{ad}_c$ and

$$A = A_1 \oplus A_0 \oplus A_{\frac{1}{2}} \text{ with}$$

$A_1 = \langle e_1 \rangle$, $A_0 = \langle e_2 \rangle$ and $A_{\frac{1}{2}} = \langle e_3 \rangle$.

This example illustrates an important general fact: $A$ satisfies the *Jordan-type* fusion law:

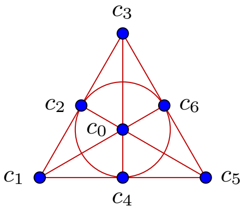
| $\star$ | $1$ | $0$ | $\frac{1}{2}$ |
|---|---|---|---|
| $1$ | $1$ | $\emptyset$ | $\frac{1}{2}$ |
| $0$ | $\emptyset$ | $0$ | $\frac{1}{2}$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1, 0$ |

# Finite simple groups

Just as prime numbers, finite simple groups are the building blocks of finite groups.

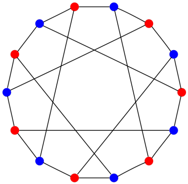
**The Classification** (1981): any finite simple group occurs either in one of 18 regular infinite series like cyclic or Lie type groups, or it is one of the 26 *sporadic groups*.

- **Example.** The linear group $L_2(7) \cong L_3(2)$ is the second smallest non-abelian simple group (of order 168). It is the group of automorphisms of the Fano plane.




The Fano plane

| points | lines |
|---|---|
| 1 | 123 |
| 2 | 145 |
| 3 | 160 |
| 4 | 246 |
| 5 | 250 |
| 6 | 340 |
| 0 | 356 |



Incidence graph (Heawood graph)

- $L_2(7)$ is a $2A$-generated subgroup of the Monster, the largest sporadic group.

# A mystery of the Monster sporadic group

- The Monster Group $\mathbb{M}$ is the largest of the *sporadic* simple groups: its order is

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \cdot \approx 8 \cdot 10^{53}$$

- 196883 is the dimension of the first nontrivial representation of the Monster.
- In 1978 McKay noticed that the Fourier coefficients of the modular function

$$\begin{aligned} j(\tau) &= \frac{(\theta_2^3(\tau) + \theta_3^3(\tau) + \theta_4^3(\tau))^3}{8\eta(\tau)^{24}} = \sum_{n=-1}^{\infty} c_n q^n \\ &= q^{-1} + 744 + (196883 + 1)q + \ldots, \qquad q = e^{2\pi i \tau} \end{aligned}$$

  are linear combinations of the character degrees of the Monster! McKay and Thompson suggested that a natural graded module for $\mathbb{M}$ might exist.
- Conway and Norton formulate 'Monstrous moonshine' (1979).
- In 1988, Frenkel, Lepowsky & Meurman *explicitly* constructed a graded **vertex operator algebra** representation

$$V^\sharp = \oplus_{n\ge 0} V_n \quad \text{of } \mathbb{M} \text{ with } V_2 = V_{\mathbb{M}}.$$

- In 1992 Borcherds finally settled the Moonshine conjecture (Fields medal 1998)

# Griess algebra

- The Monster was constructed in 1982 by Griess as a group of automorphisms of a **non-associative commutative unital algebra** $V_{\mathbb{M}}$ of dimension $1 + 196883$ over $\mathbb{R}$, now called the Griess algebra.
- The Griess algebra admits a natural invariant associating bilinear form, i.e.

$$\langle xy, z\rangle = \langle x, yz\rangle, \quad \forall x, y, z \in V_{\mathbb{M}}$$

- The Monster is generated by its involutions (the so-called (2A)-invoutions).
- There is a natural bijection between the (2A)-invoultions and a subset of idempotents (*axes*) in $V_{\mathbb{M}}$
- Any axis induces an orthogonal decomposition $A = A_1 \oplus A_0 \oplus A_{\frac{1}{4}} \oplus A_{\frac{1}{32}}$ with the **Ising fusion laws**:

| $\star$ | $1$ | $0$ | $\frac{1}{4}$ | $\frac{1}{32}$ |
|---|---|---|---|---|
| $1$ | $1$ | $\emptyset$ | $\frac{1}{4}$ | $\frac{1}{32}$ |
| $0$ | $\emptyset$ | $0$ | $\frac{1}{4}$ | $\frac{1}{32}$ |
| $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $1, 0$ | $\frac{1}{32}$ |
| $\frac{1}{32}$ | $\frac{1}{32}$ | $\frac{1}{32}$ | $\frac{1}{32}$ | $1, 0, \frac{1}{4}$ |

| $\star$ | $+$ | $+$ | $+$ | $-$ |
|---|---|---|---|---|
| $+$ | $+$ | $+$ | $+$ | $-$ |
| $+$ | $+$ | $+$ | $+$ | $-$ |
| $+$ | $+$ | $+$ | $+$ | $-$ |
| $-$ | $-$ | $-$ | $-$ | $+$ |

# Griess algebra as a Majorana algebra

| $\star$ | $1$ | $0$ | $\frac{1}{4}$ | $\frac{1}{32}$ |
|---|---|---|---|---|
| $1$ | $1$ | $\emptyset$ | $\frac{1}{4}$ | $\frac{1}{32}$ |
| $0$ | $\emptyset$ | $0$ | $\frac{1}{4}$ | $\frac{1}{32}$ |
| $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $1, 0$ | $\frac{1}{32}$ |
| $\frac{1}{32}$ | $\frac{1}{32}$ | $\frac{1}{32}$ | $\frac{1}{32}$ | $1, 0, \frac{1}{4}$ |

| $\star$ | $+$ | $+$ | $+$ | $-$ |
|---|---|---|---|---|
| $+$ | $+$ | $+$ | $+$ | $-$ |
| $+$ | $+$ | $+$ | $+$ | $-$ |
| $+$ | $+$ | $+$ | $+$ | $-$ |
| $-$ | $-$ | $-$ | $-$ | $+$ |

- $A_+ := A_1 \oplus A_0 \oplus A_{\frac{1}{4}}$ is a subalgebra
- There is a natural grading $A = A_+ \oplus A_-$ with $A_- := A_{\frac{1}{32}}$
- A key element of the construction: the Majorana involution

$$x \to x^\tau := x_1 + x_0 + x_{\frac{1}{4}} - x_{\frac{1}{32}}$$

- Furthermore, if $x_{\frac{1}{32}} = 0$ then $x \to x^\sigma := x_1 + x_0 - x_{\frac{1}{4}}$ is also an algebra involution.

The above algebra structure determines the Monster as an isomorphism group.

# G-NA-VOA project

- In his talks on G-NA-VOA project, Robert Griess pointed out that a good theory of commutative nonassociative algebras generalizing the Monster algebra is highly relevant to developing a unified framework of finite simple groups, especially the sporadic groups.
- A starting point is the commutative nonassociative algebra $V_2 = V_{\mathbb{M}}$ in the VOA $V^\sharp$.
- Via foundational work of M.Miyamoto (1995) and later by A.A.Ivanov (2007), the Griess algebra $V_2$ has been understood as a commutative non-associative algebra generated by a finite set of *primitive* idempotents (i.e. the dimension of the $1$-eigenspace is one) satisfying certain axioms, the so-called *Majorana algebras*.
- An algebra is *Majorana* if it possesses a positive definite associating bilinear form $\langle x, y \rangle$ and it is generated (as an algebra) by a finite set of idempotents with the $(1, 0, \frac{1}{4}, \frac{1}{32})$-Peirce decomposition and the Ising fusion laws. Some further (somewhat subtle) axioms like the Norton inequality are also required.
- (Ivanov, *The Monster Group and Majorana Involutions*, CTM, 2009) proves that the Griess algebra of the Monster is a Majorana algebra and that the action of the Monster on its algebra realizes a Majorana representation of the Monster.

# Axial algebras

- A further, radical, step was taken by J. Hall, S. Shpectorov & F. Rehren (2015), they put the previous work into a natural context of *axial algebras*.
- An *axial algebra* over the field $\mathbb{F}$ is a commutative algebra generated by idempotents whose adjoint action has the same multiplicity-free minimal polynomial.

  Axial algebras include Jordan algebras, the Matsuo algebras for groups of 3-transpositions, as well as the 196884-dimensional Griess algebra.
- Recall that $V_{\mathbb{M}}$ is generated by $(2A)$-idempotents satisfying the Ising fusion laws. J. Hall, S. Shpectorov and F. Rehren, (2015) completely characterized axial algebras satisfying the Jordan type fusion laws for $\eta \neq \frac{1}{2}$

| $\star$ | $1$ | $0$ | $\eta$ |
|---|---|---|---|
| $1$ | $1$ | $\emptyset$ | $\eta$ |
| $0$ | $\emptyset$ | $0$ | $\eta$ |
| $\eta$ | $\eta$ | $\eta$ | $1, 0$ |

  The case $\eta = \frac{1}{2}$ is singular (contains also Jordan algebras) and is not settled yet.

# How incident is a particular Peirce spectrum?

We have seen that

- for Jordan algebras $\mathbf{spec}(c) \subset \{1, 0, \frac{1}{2}\}$
- while for the Griess algebra $\mathbf{spec}(c) \subset \{1, 0, \frac{1}{4}, \frac{1}{32}\}$

(Krasnov Ya., V.T., 2018) proved that a *generic* algebra does not contain $\frac{1}{2}$ in its spectrum. On the other hand, we have

## Theorem (V.T., 2018, submitted)

Let $A$ be a finite dimensional commutative nonassociative algebra over a field of characteristic $\neq 2, 3$ and let $A$ satisfy a nontrivial identity $P(z) = 0$. Then

(A) $\frac{1}{2} \in \mathbf{spec}(c)$ for any idempotent $c \neq 0$;

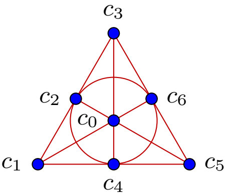
(B) if $c$ is semi-simple and $\lambda$ is single root of the Peirce polynomial $\varrho(P, t)$ then

$$A_\lambda A_{\frac{1}{2}} \subset A_\lambda^\perp := \bigoplus_{\substack{\nu \in \mathbf{spec}(c) \\ \nu \neq \lambda}} A_\nu.$$

In particular, $A_{\frac{1}{2}} A_{\frac{1}{2}} \subset A_{\frac{1}{2}}^\perp$.

# A toy example: the Harada algebra

Motivated by Griess' construction of the Monster, the same year Koichiro Harada defined a 6-dimensional algebra $H$ whose automorphism group is $L_2(7)$ as follows:




$c_1, \ldots, c_6$ generate $H$ as a vector space with multiplication

$$c_i c_j = \begin{cases} c_i & \text{if } i = j \\ c_{i \wedge j}, & \text{if } i \neq j, \end{cases}$$

and an additional identity

$$c_0 + c_1 + \ldots + c_6 = 0.$$

For example, $c_1 c_2 = c_{1 \wedge 2} = c_3$.

- Each line is a subalgebra!
- Any automorphism $g \in \mathrm{Aut}(H)$ preserves idempotents: $x^2 = x \Rightarrow g(x)^2 = g(x)$.
- Hence, $\mathrm{Aut}(H)$ acts on $\Phi$ by permutations.
- Since $\{c_i\}$ generates $H$, the group $\mathrm{Aut}(H)$ acts faithfully on $\Phi$.
- Since each line is a subalgebra, $\mathrm{Aut}(H)$ act naturally on lines.

This proves that

$$\mathrm{Aut}(H) = \mathrm{Aut}(\Phi) = L_2(7).$$

# The Harada algebra

- $\Phi$ has totally

$$\#\mathrm{Idem}(H) = 2^{\dim H} - 1 = 63 = 7 + 7 + 21 + 28 \quad \text{nonzero idempotents.}$$

- There are exactly five orbits of the $\mathrm{Aut}(H)$-action on $\mathrm{Idem}(H)$:
  - 7 points: $c_i$
  - 7 lines: $\frac{1}{3}(c_{i\wedge j} + c_i + c_j)$,
  - 21 flags: $\frac{1}{3}(2c_{i\wedge j} - c_i - c_j)$;
  - 28 anti-flags: $-3(c_{i\wedge j} + c_i + c_j) - 5c_m$.

**Theorem** (Ya. Krasnov, T., 2018, in preparation) Idempotents in each orbit have the same spectrum and the only graded (Ising type) fusion laws occur for flags:

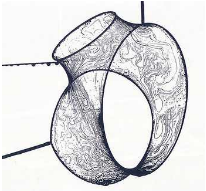
| $\star$ | $1$ | $0$ | $-1$ | $\frac{4}{3}$ | $-\frac{2}{3}$ |
|---|---|---|---|---|---|
| $1$ | $1$ | $\bullet$ | $-1$ | $\frac{4}{3}$ | $-\frac{2}{3}$ |
| $0$ | $\bullet$ | $0$ | $\bullet$ | $\frac{4}{3}$ | $-\frac{2}{3}$ |
| $-1$ | $-1$ | $\bullet$ | $1$ | $\bullet$ | $-\frac{2}{3}$ |
| $\frac{4}{3}$ | $\frac{4}{3}$ | $\frac{4}{3}$ | $\bullet$ | $1, 0$ | $-\frac{2}{3}$ |
| $-\frac{2}{3}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ | $1, 0, \frac{1}{4}$ |

# Algebras of cubic minimal cones

A **minimal surface** minimizes locally the area functional.

Geometrically, this means that the **mean curvature** $= 0$.

Analytically, a solution of certain nonlinear PDE.



A **minimal cone** is a typical singularity of a minimal surface. The most of known minimal cones are algebraic, i.e. zero level sets of a homogeneous polynomial $u \in \mathbb{R}[x_1, \ldots, x_n]$.

Degree 2 minimal cones are well-known. An example is the Clifford-Simon cone given by

$$u(x) := (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$$

which is also known as the norm for split octonions. This cone played a crucial role in the solution of the famous Bernstein problem and counterexamples in higher dimensions.

# Hsiang minimal cones

We shall concern with *cubic* minimal cones, i.e. cones of the smallest nontrivial degree 3.

**Hsiang's problem (1967).** How to characterize cubic minimal cones? Can one at least characterize all cubic polynomial solutions of

$$|Du|^2\Delta u - \tfrac{1}{2}\langle Du, D|Du|^2\rangle = \lambda|x|^2 u,$$

or, more explicitly,

$$(u_{x_1}^2 + \ldots + u_{x_n}^2)(u_{x_1x_1} + \ldots + u_{x_nx_n}) - \sum_{i,j=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j} = \lambda(x_1^2 + \ldots + x_n^2)\cdot u(x)$$

# Some explicit examples

- $u = \mathrm{Re}(z_1 z_2) z_3$, the triality polynomials in $\mathbb{R}^{3d}$, where $z_i \in \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$

- $u(x) = \begin{vmatrix} \frac{1}{\sqrt{3}}x_1 + x_2 & x_3 & x_4 \\ x_2 & \frac{-2}{\sqrt{3}}x_1 & x_5 \\ x_4 & x_5 & \frac{1}{\sqrt{3}}x_1 - x_2 \end{vmatrix}$

  the generic norm in the Jordan algebra of $3 \times 3$ symmetric matrices over $\mathbb{R}$

- $u(x) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{vmatrix}$

  the generic norm in the Jordan algebra of $4 \times 4$ symmetric *traceless* matrices over $\mathbb{R}$

Thus, Hsiang cubics are nicely encoded by certain algebraic structures. Which ones?

# A metrized algebra approach

Let $u$ be a cubic form on a vector space $V$ over $\mathbb{F}$ with an inner product $\langle x, y\rangle$. Linearize $u$ to get a **symmetric trilinear** form $\widetilde{u}(x,y,z)$ such that

$$\widetilde{u}(x,x,x) = 6u(x)$$

Define a multiplication $(x,y) \to xy$ on $V$ by the duality:

$$\widetilde{u}(x,y,z) = \langle xy, z\rangle, \qquad \forall z \in V.$$

Thus obtained algebra $V(u)$ is **commutative** and metrized:

$$\langle xy, z\rangle = \langle x, yz\rangle$$

and the cubic form is recovered by $u(x) = \frac{1}{6}\langle x, x^2\rangle$. We also have

- $x^2 = 2\,\mathrm{grad}\, u(x)$
- $x$ is an idempotent in $V$ iff $kx$ is a stationary point of $u$ ($x \parallel \mathrm{grad}\, u(x)$).
- $xy = \mathrm{Hess}\, u(x) y \quad \Rightarrow \quad \mathrm{ad}_x = \mathrm{Hess}\, u(x)$
- $\mathrm{ad}_x$ is **self-adjoint**: $\langle \mathrm{ad}_x\, y, z\rangle = \langle y, \mathrm{ad}_x\, z\rangle$
- Any idempotent is semisimple!

# Algebras of minimal cones

In the above setup,

$$\langle Du, Du\rangle \Delta u - \langle Du, \mathrm{Hess}\,(u) Du\rangle = \lambda |x|^2 u$$

yields

$$\langle \frac{x^2}{2}, \frac{x^2}{2}\rangle \,\mathrm{tr}(\mathrm{ad}_x) - \frac{1}{2}\langle \frac{x^2}{2}, \frac{x^3}{2}\rangle = \frac{\lambda}{6}\langle x, x\rangle\langle x, x^2\rangle.$$

One can show that $u$ is harmonic (non-trivial!), i.e.

$\mathrm{ad}_x$ is trace less!

This implies that to any cubic cone one can attach a commutative metrized algebra with

$$\begin{cases} \langle x^2, x^3\rangle & = \langle x, x\rangle\langle x^2, x\rangle \\ \mathrm{tr}(\mathrm{ad}_x) & = 0. \end{cases} \quad (1)$$

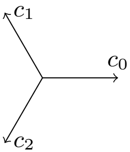
Conversely: if $A$ satisfies (1) then $u(x) = \frac{1}{6}\langle x, x^2\rangle$ generates a minimal cone.

We arrive at classification of all algebras satisfying (1). We call these **Hsiang algebras**.

# Two basic examples in dimensions 2 and 3

**Example 1.** Let $A$ be the algebra on $\mathbb{R}^2$ generated by **idempotents** $c_0, c_1, c_2$ with

$$\boxed{c_0 + c_1 + c_2 = 0}$$




Then for any triple $\{i, j, k\} = \{1, 2, 3\}$

$$c_k = c_k^2 = (-c_i - c_j)^2 = c_i + c_j + 2c_ic_j = -c_k + 2c_ic_j$$

$$\Rightarrow \quad c_ic_j = c_k \quad \Rightarrow \quad c_k(c_i - c_j) = -1(c_i - c_j).$$

This implies that $A$ is a Hsiang algebra, $A = A_1 \oplus A_{-1}$ with $\mathbb{Z}_2$-graded fusion laws

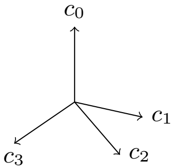
| $\star$ | $1$ | $-1$ |
|---|---|---|
| $1$ | $1$ | $-1$ |
| $-1$ | $-1$ | $1$ |

The **minimal cone** is given by $x_1^2x_2 = 0$, i.e. pair of two orthogonal lines in $\mathbb{R}^2$

# Two basic examples in dimensions 2 and 3

**Example 2.** Let $A$ be generated by **idempotents** $c_0, c_1, c_2, c_3$ in $\mathbb{R}^3$ subject to

$$\boxed{(c_i + c_j)^2 = 0 \qquad \text{for } i \neq j}$$




Then similarly to the above, one easily verifies that

$$A = A_1 \oplus A_{-\frac{1}{2}}, \quad \dim A_1 = 1, \quad \dim A_{-\frac{1}{2}} = 2.$$

Then $A$ is a Hsiang algebra with fusion laws

| $\star$ | $1$ | $-\frac{1}{2}$ |
|---|---|---|
| $1$ | $1$ | $-\frac{1}{2}$ |
| $-\frac{1}{2}$ | $-\frac{1}{2}$ | $1, -\frac{1}{2}$ |

After a *rank one perturbation* $A$ becomes a Jordan algebra of Clifford type.
The **minimal cone** is given by $x_1 x_2 x_3 = 0$, i.e. the triple of coordinate planes in $\mathbb{R}^3$.

# Generalization

**Definition.** A commutative algebra $V$ with associating form $\langle x, y\rangle$ is called polar if

- there is a $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$,
- $V_0 V_0 = 0$,
- $x(xy) = |x|^2 y$ for $x \in V_0$ and $y \in V_1$.

Any polar algebra is Hsiang!

**Remark.** Can be simply constructed using **symmetric Clifford systems**, i.e. symmetric matrices

$$A_i^2 = I, \qquad A_i A_j + A_j A_i = 0, \quad i \neq j$$

This yields an obstruction:

$$q \leq 1 + \rho(p),$$

where $\rho(m) = 8a + 2^b$, if $m = 2^{4a+b} \cdot \mathrm{odd}$, $0 \leq b \leq 3$ is the **Hurwitz-Radon function**.

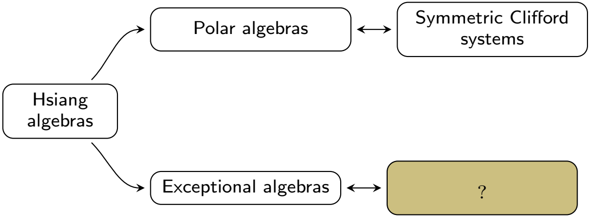
**Definition.** A Hsiang algebra $V$ isomorphic to a polar algebra is of **Clifford type**; otherwise it is **exceptional**.

# Exceptional Hsiang algebras

Any commutative **pseudocomposition algebra**, i.e. an algebra with

$$x^3 = |x|^2 x,$$
$$\operatorname{tr} \operatorname{ad}_x = 0$$

is an *exceptional* Hsiang algebra.




We prove that there are only finitely many exceptional Hsiang algebras!

# Regular vs Exceptional

- Many classification results reminiscence the famous Cartan-Killing classification: there are several 'nice' series and finitely many exceptional (sporadic) elements.
  - the $A$, $B$, $C$, $D$-series of simple Lie groups and the exceptional Lie groups $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$;
  - special Jordan algebras (obtained from associative algebras) vs the Albert 27-dimensional exceptional Jordan algebra
  - the classification of finite simple groups
- The fundamental question is why exceptional (sporadic) objects do really exist?

# Finiteness of exceptional Hsiang algebras

**The Peirce decomposition** Let $c$ be an idempotent. Then

$$A = A_1 \oplus A_{-1} \oplus A_{-\frac{1}{2}} \oplus A_{\frac{1}{2}}$$

where $A_1 = \mathbb{R}c$ (i.e. $c$ is primitive!) satisfying the Hsiang algebras fusion laws:

| $\star$ | $1$ | $-1$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
|---|---|---|---|---|
| $1$ | $1$ | $-1$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| $-1$ | $-1$ | $1$ | $\frac{1}{2}$ | $-\frac{1}{2}, \frac{1}{2}$ |
| $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $1, -\frac{1}{2}$ | $-1, \frac{1}{2}$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}, \frac{1}{2}$ | $-1, \frac{1}{2}$ | $1, -1, -\frac{1}{2}$ |

- Two distinguished subalgebras: $A_1 \oplus A_{-1}$ (carries a hidden Clifford algebra structure) and $A_1 \oplus A_{-\frac{1}{2}}$ (carries a hidden rank 3 Jordan algebra structure)
- Primitive idempotents $w$ in the hidden Jordan algebra ($w \bullet w = w$) are exactly 2-nilpotents in $V$ ($w^2 = w$) with the fusion rules

# Finiteness of exceptional Hsiang algebras

**Main Theorem (V.T., 2014, 2016)**

(i) Given an idempotent $c \in A$, the new algebra on $\Lambda_c = (A_1 \oplus A_{-\frac{1}{2}}, \bullet)$ with

$$x \bullet y = \frac{1}{2} xy + \langle x, c\rangle y + \langle y, c\rangle x - 2\langle xy, c\rangle c.$$

is a Euclidean Jordan algebra of rank 3 with unit $c^* = 2c$.

(ii) $A$ is exceptional if and only if the Jordan algebra $\Lambda_c$ is simple.

(iii) There holds

$$\dim A_{-1} - 1 \leq \rho(\dim A_{-1} + \dim A_{-\frac{1}{2}} - 1),$$

where $\rho$ is the Hurwitz-Radon function.

(iv) There are at most 16 possible classes of exceptional Hsiang algebras:

| $\dim A$ | 2 | 5 | 8 | 14 | 26 | 9 | 12 | 21 | 15 | 18 | 24 | 30 | 42 | 27 | 30 | 54 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\dim A_{-1}$ | 1 | 2 | 3 | 5 | 9 | 0 | 1 | 4 | 0 | 1 | 3 | 5 | 9 | 0 | 1 | 1 |
| $\dim A_{-1/2}$ | 0 | 0 | 0 | 0 | 0 | 5 | 5 | 5 | 8 | 8 | 8 | 8 | 8 | 14 | 14 | 26 |

unsettled (columns 24, 30, 42)

# Realization of cubic cones via Jordan algebras

**Theorem (Existence)**

Denote by $V(u)$ the algebra on a vector inner product space $V$ generated by the cubic form $u$. Then $A = V(u)$ where

- $\dim A_{-\frac{1}{2}} = 0$, $\dim A_{-1} = d+1$, and $u = \frac{1}{6}\langle z, z^2\rangle$, $V = \mathscr{H}_3(\mathbb{K}_d) \ominus \mathbb{R}e$, $d = 0, 1, 2, 4, 8$.
- $\dim A_{-1} = 0$, $\dim A_{-\frac{1}{2}} = \frac{3}{2}d + 2$, and $u = \frac{1}{12}\langle z^2, 3\bar{z} - z\rangle$, where $z \to \bar{z}$ is the natural involution on $V = \mathscr{H}_3(\mathbb{K}_d)$, $d = 2, 4, 8$.
- $\dim A_{-1} = 1$, $\dim A_{-\frac{1}{2}} = 3d + 2$, and $u(z) = \mathrm{Re}\langle z, z^2\rangle$, where $z \in V = \mathscr{H}_3(\mathbb{K}_d) \otimes \mathbb{C}$, $d = 1, 2, 4, 8$.
- $\dim A_{-1} = 4$ and $\dim A_{-\frac{1}{2}} = 5$, where $u = \frac{1}{6}\langle z, z^2\rangle$ on $\mathscr{H}_3(\mathbb{K}_8) \ominus \mathscr{H}_3(\mathbb{K}_1)$

# Epilogue: Nonassociative algebras and singular solutions

- Evans, Crandall, Lions, Jensen, Ishii: If $\Omega \subset \mathbb{R}^n$ is bounded with $C^1$-boundary, $\phi$ continuous on $\partial\Omega$, $F$ uniformly elliptic operator then the Dirichlet problem

$$F(D^2u) = 0, \quad \text{in} \quad \Omega$$
$$u = \phi \quad \text{on} \quad \partial\Omega$$

  has a unique **viscosity solution** $u \in C(\Omega)$;
- Krylov, Safonov, Trudinger, Caffarelli, early 80's: the solution is always $C^{1,\varepsilon}$
- Nirenberg, 50's: if $n = 2$ then $u$ is classical ($C^2$) solution
- Nadirashvili, Vlăduț, 2007-2011: if $n \geq 12$ then there are solutions which are not $C^2$.

**Theorem (Nadirashvili-V.T.-Vlăduț**, 2012, 2015) *The function* $w(x) := \frac{u_1(x)}{|x|}$ *where*

$$u_1(x) = x_5^3 + \tfrac{3}{2}x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \tfrac{3\sqrt{3}}{2}x_4(x_2^2 - x_1^2) + 3\sqrt{3}x_1x_2x_3,$$

*is a singular viscosity solution of the uniformly elliptic Hessian equation*

$$(\Delta w)^5 + 2^8 3^2 (\Delta w)^3 + 2^{12} 3^5 \Delta w + 2^{15} \det D^2(w) = 0,$$

*This also gives the best possible dimension ($n = 5$) where* homogeneous *order 2 real analytic functions in* $\mathbb{R}^n \setminus \{0\}$.

# Some important questions remain unanswered

How incident (important, relevant) that the certain *commutative non-associative* algebraic structures coming from finite simple groups, geometry of minimal cones and PDEs (truly viscosity solutions)

- have a distinguished Peirce spectrum
- have distinguished (in particular, graded) fusion rules
- are axial
- are metrized (i.e. carrying an associating symmetric bilinear form)?

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**THANK YOU FOR YOUR ATTENTION!**