Undeformed commutators in q-deformed Heisenberg algebras

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Definition

Let $\mathbb{F}$ be a field, and let $q \in \mathbb{F}$. The *q-deformed Heisenberg algebra* $\mathcal{H}(q)$ is the unital associative algebra over $\mathbb{F}$ that has a presentation by generators A, B and relation

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Goal: Study the Lie subalgebra $\mathfrak{L}(q)$ of $\mathcal{H}(q)$ generated by A, B.

Preliminaries

Some notation:

$$\begin{aligned}\mathbb{F} &= \text{arbitrary field,}\\ \mathbb{N} &= \text{set of all nonnegative integers,}\\ \mathbb{Z}^+ &= \text{set of all positive integers,}\end{aligned}$$

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Some preliminary notions:

1. Let $t \in \mathbb{N}$. By a *word* of *length* t on $\mathcal{X}$ we mean a finite sequence of the form

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4. Given words $X_1 X_2 \cdots X_s$ and $Y_1 Y_2 \cdots Y_t$ on $\mathcal{X}$, their *concatenation product* is the word
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2. The multiplication operation in $\mathbb{F}\langle\mathcal{X}\rangle$ is completely determined by the concatenation product of words on $\mathcal{X}$.
3. Let $f_1, f_2, \ldots, f_k \in \mathbb{F}\langle\mathcal{X}\rangle$, and let $\mathcal{J}$ be the (two-sided) ideal of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by $f_1, f_2, \ldots, f_k$. Denote the elements of $\mathcal{X}$ by $G_1, G_2, \ldots, G_n$. Then the algebra defined by a presentation having generators $G_1, G_2, \ldots G_n$ and relations $f_1 = 0, f_2 = 0, \ldots, f_k = 0$ is precisely the quotient algebra $\mathbb{F}\langle\mathcal{X}\rangle/\mathcal{J}$.

Example

Let $q \in \mathbb{F}$, and set $\mathcal{X} = \{A, B\}$. Denote by $\mathcal{J}$ the ideal of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by $AB - qBA - I$. Then $\mathcal{H}(q) = \mathbb{F}\langle\mathcal{X}\rangle / \mathcal{J}$.

The free algebra $\mathbb{F}\langle\mathcal{X}\rangle$ and the free Lie algebra on $\mathcal{X}$

1. The free Lie algebra on $\mathcal{X}$ (or the set of all Lie polynomials in $\mathcal{X}$) is the Lie subalgebra $\mathcal{L} := \mathcal{L}_{\mathcal{X}}$ of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by $\mathcal{X}$.

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Proposition

With reference to above notation, given the canonical map $\varphi : \mathbb{F}\langle\mathcal{X}\rangle \to \mathbb{F}\langle\mathcal{X}\rangle/\mathcal{J}$, and a basis $\mathcal{B}$ of $\mathcal{L}$ then a spanning set for the Lie algebra $\mathcal{L}/(\mathcal{J}\cap\mathcal{L})$ consists of vectors of the form

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2. Is it possible to use consequences of the relation $AB - qBA - I = 0$ to reduce the corresponding spanning set into a basis?
3. Given a basis for $\mathfrak{L}(q)$, compute the commutator table.

A basis for $\mathcal{L}$ consisting of regular words on A, B

The following notions are from the formulation given in (Ufnarovskij, 1995). For a full discussion on regular words on n generators, refer to arXiv:1709.02612 (Section 2).

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A word on A, B is regular if it is a generator or if, with respect to $\lhd$, it is strictly greater than any of its cyclic permutations.

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We define $[\![A]\!] := A$, and $[\![B]\!] := B$ as the *regular nonassociative words* on A, B of length 1.

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We define $[\![A]\!] := A$, and $[\![B]\!] := B$ as the *regular nonassociative words* on A, B of length 1. Given $t \in \mathbb{Z}^+$, suppose that all regular nonassociative words on A, B of lengths strictly less t have been defined.

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The regular nonassociative words on A, B form a basis for the free Lie algebra on A, B.

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5. $[\![B^3A^4BA]\!] = [B, [[\![B^2A^4]\!], [\![BA]\!]]] = \cdots$
6. $[\![BAB^3A^4]\!]$ is undefined.

Effect of the relation $AB - qBA - I = 0$ on the regular words

To know which of the regular nonassociative words on A, B (spanning set elements of $\mathfrak{L}(q)$) can be removed and obtain a maximal linearly independent set, the following result was consequential.

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Lemma (Hellström and Silvestrov, 2005)

The following vectors form a basis for $\mathcal{H}(q)$.

$$[A,B]^k, \quad [A,B]^k A^l, \quad B^l [A,B]^k, \quad (k \in \mathbb{N}, l \in \mathbb{Z}^+). \tag{1}$$

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How is the product of any two basis elements in (1) expressible as a linear combination of (1)?

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$$\begin{aligned} [A,B]^k B^l &= q^{kl} B^l [A,B]^k\,, \\ A^l [A,B]^k &= q^{kl} [A,B]^k A^l, \\ B^l A^l &= q^{-\binom{l}{2}} (q-1)^{-l} \sum_{i=0}^{l} (-1)^{l-i} q^{\binom{l-i}{2}} \binom{l}{i}_q [A,B]^i\,, \\ A^l B^l &= (q-1)^{-l} \sum_{i=0}^{l} (-1)^{l-i} q^{\binom{i+1}{2}} \binom{l}{i}_q [A,B]^i\,. \end{aligned}$$

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\end{aligned}$$

where the expression $\binom{l}{i}_q$ is as described in the following: Given $n \in \mathbb{N}$, let $\{n\}_q := 1 + q + q^2 + \cdots + q^{n-1}$, and $\{n\}_q! := \{n\}_q \{n-1\}_q \cdots \{1\}_q$. If $k \in \mathbb{N}$ with $k \leq n$, we define the number $\binom{n}{k}_q$ as 1 if $k \in \{0, n\}$, or as the expression $\frac{\{n\}_q!}{\{k\}_q! \{n-k\}_q!}$, otherwise.

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Some consequences of the simple relation $AB - qBA = I$.

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Relations (2) to (4) are also from (Hellström and Silvestrov, 2005), while (5) was proven using routine computations and arguments (arXiv:1709.02612, Proposition 3.3).

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Theorem (Cantuba, 2017)

If q is nonzero and is not a root of unity, then the following vectors form a basis for $\mathfrak{L}(q)$.

$$A, B, [\![BA]\!], \left[\!\!\left[(BA)^k BA^{l+1}\right]\!\!\right],$$
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(details of the commutator table in arXiv 1709.02612, Section 5)

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$\mathcal{H}(q) = \mathfrak{L}(q) \oplus \mathrm{Span}\ \{I, A^2, B^2, A^3, B^3, \ldots\}$.

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The Lie algebra $\mathfrak{L}(q)$ is a Lie ideal of $\mathcal{H}(q)$. The resulting quotient Lie algebra as an infinite basis consisting of mutually commuting elements.

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The Fairlie-Odesskii algebra:
$U_q'(\mathfrak{so}_3)$:= the algebra with generators I_1, I_2, I_3 and relations

$$\begin{aligned} q^{\frac{1}{2}} I_1 I_2 - q^{-\frac{1}{2}} I_2 I_1 &= I_3, \\ q^{\frac{1}{2}} I_2 I_3 - q^{-\frac{1}{2}} I_3 I_2 &= I_1, \\ q^{\frac{1}{2}} I_3 I_1 - q^{-\frac{1}{2}} I_1 I_3 &= I_2. \end{aligned}$$

The original setting for this type of Lie algebra problem

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(same type of Lie algebra problems for the above algebras)

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Thank you for your attention!!!