On non-associative Weyl algebras and a Hilbert's basis theorem

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Introduction

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The former have a known non-associative structure: what about *hom-associative Ore extensions*?

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Definition (Hom-associative algebra)**.** A *hom-associative algebra* over an associative, commutative, and unital ring R, is a triple $(M, \cdot, \alpha)$ consisting of an R-module M, a binary operation $\cdot\colon M \times M \to M$ linear in both arguments, and a linear map $\alpha\colon M \to M$ satisfying, for all $a, b, c \in M$,

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Definition (Hom-ideal)**.** A *right (left) hom-ideal* of a hom-associative algebra is a right (left) algebra ideal I such that $\alpha(I) \subseteq I$. If I is both a left and a right hom-ideal, we simply call it a *hom-ideal*.

Definition (Weakly unital algebra)**.** Let $(M, \cdot, \alpha)$ be a hom-associative algebra. If for all $a \in M$, $e \cdot a = a \cdot e = \alpha(a)$ for some $e \in M$, we say that $(M, \cdot, \alpha)$ is *weakly unital* with *weak unit* e.

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Proposition (Weak unitalization [BRS18])**.** Any multiplicative hom-associative algebra can be embedded in a multiplicative, weakly unital hom-associative algebra.

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Proposition (The Yau twist [Yau09])**.** Let A be a unital, associative algebra, α an algebra endomorphism on A and define $*\colon A \times A \to A$ by $a * b := \alpha(a \cdot b)$ for all $a, b \in A$. Then $(A, *, \alpha)$ is a weakly unital hom-associative algebra with weak unit 1_A.

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Given a non-associative, non-unital ring R with left R-additive maps $\delta\colon R \to R$ and $\sigma\colon R \to R$, by a *non-associative, non-unital Ore extension* of R, $R[x; \sigma, \delta]$, we mean $\left\{\sum_{i\in\mathbb{N}} a_i x^i\right\}$, finitely many $a_i \in R$ non-zero,

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$$\sum_{i\in\mathbb{N}} a_i x^i + \sum_{i\in\mathbb{N}} b_i x^i := \sum_{i\in\mathbb{N}} (a_i + b_i) x^i, \quad a_i, b_i \in R,$$

two polynomials being equal iff their coefficients are, $\forall a, b \in R$,

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Here π_i^m is the sum of all $\binom{m}{i}$ compositions of i copies of σ and $m-i$ copies of δ. For example $\pi_0^0 = \mathrm{id}_R$ and $\pi_1^2 = \sigma\circ\delta + \delta\circ\sigma$. Imposing distributivity of the multiplication over addition makes $R[x;\sigma,\delta]$ a ring.

For instance,

$$ax^0 \cdot bx^0 = \sum_{i \in \mathbb{N}} \big(a \cdot \pi_i^0(b)\big) x^{i+0} = (a \cdot b)x^0, \text{ so } R \cong Rx^0,$$

For instance,

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Remark. If R contains a unit, we write x for the formal sum $\sum_{i \in \mathbb{N}} a_i x^i$ with $a_1 = 1$ and $a_i = 0$ when $i \neq 1$. It does not necessarily make sense to think of x as an element of the non-associative Ore extension if R is not unital.

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Definition (σ-derivation)**.** Let R be a non-unital, non-associative ring where σ is an endomorphism and δ an additive map on R. Then δ is called a *σ-derivation* if $\delta(a\cdot b) = \sigma(a)\cdot\delta(b) + \delta(a)\cdot b$ holds for all $a, b \in R$. If $\sigma = \mathrm{id}_R$, δ is a *derivation*.

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Definition (Homogeneous map)**.** Let $R[x;\sigma,\delta]$ be a non-associative, non-unital Ore extension of a non-associative, non-unital ring R.

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Proposition ([BRS18])**.** Let $R[x;\sigma,\delta]$ be a hom-associative, non-unital Ore extension of a hom-associative, non-unital ring R, with the additive map $\alpha\colon R \to R$ extended homogeneously to $R[x;\sigma,\delta]$. Then, for all $a, b, c \in R$,

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EXAMPLES

Examples

Example (Hom-associative quantum planes [BRS18]). The quantum planes $Q_q(K) := K\langle x, y\rangle / \langle x \cdot y - qy \cdot x\rangle$ can be presented as $K[y][x; \sigma, 0]$ where K is a field of characteristic zero and σ the unital K-algebra automorphism of $K[y]$ such that $\sigma(y) = qy$ and $q \in K^{\times}$ (multiplicative group of nonzero elements).

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This is a k-family $\{Q_q^k(K)\}_{k\in K}$ of weakly-unital, hom-associative Ore extensions $Q_q^k(K) := (Q_q(K), *, \alpha_k)$ with weak unit 1_K, where for instance
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We call these *hom-associative quantum planes*, the commutation relation $x \cdot y = qy \cdot x$ becoming $x * y = kqy * x$, including the associative quantum planes in $k = 1_K$.

Example (Hom-associative Weyl algebras [BRS18])**.** Consider the first Weyl algebras $W(K) := K\langle x, y\rangle/\langle x \cdot y - y \cdot x - 1_K\rangle$ as $K[y][x; \mathrm{id}_{K[y]}, \delta]$, where K is a field of characteristic zero, and $\delta = \frac{\mathrm{d}}{\mathrm{d}y}$. An algebra endomorphism α_k on $K[y]$ commutes with δ (and $\mathrm{id}_{K[y]}$) iff $\alpha_k(y) = y + k,\ k \in K$, and $\alpha_k(1_K) = 1_K$.

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We call these *hom-associative Weyl algebras*, the commutation relation $x \cdot y - y \cdot x = 1_K$ becoming $x * y - y * x = 1_K$.

Proposition ([Bäc18]). The following hold in $W^k(K)$:
$\left[x, p(y)\right]_* = p'(y + k),$ for any polynomial $p(y)$ in y.
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Note. The Weyl algebras are *associatively* formally rigid.

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(N2) Any nonempty family of right (left) hom-ideals of R has a maximal element.

(N3) Any right (left) hom-ideal of R is finitely generated.

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Corollary (Hilbert's basis theorem for non-associative rings [BR18])**.** Let R be a unital, non-associative ring, σ an automorphism and δ a σ-derivation on R. If R is right (left) noetherian, then so is $R[x;\sigma,\delta]$.

Example (Octonionic Weyl algebra [BR18])**.** Denote by $\mathbb{O}$ the octonions; $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$. We define the *octonionic Weyl algebra* as $\mathbb{O}[y][x; \mathrm{id}_{\mathbb{O}[y]}, \delta]$, where $\delta := \frac{\mathrm{d}}{\mathrm{d}y}$; hence $x \cdot y - y \cdot x = 1_{\mathbb{O}}$.

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Proposition ([BR18])**.** The octonionic Weyl algebra is noetherian.

Thank you!