$q-$ based derivation Levi Civita Connection on a noncommutative sphere

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Introduction

- The aim of this work is to understand and define $q-$deformed Levi-Civita Connection on the quantum $3-$sphere.
- In noncommutative geometry, one is interested to find the generalizations of geometrical objects such as vector fields, differential forms, etc so that the classical objects are simply particular case.
- There are different approaches to noncommutative geometry with special consideration as starting points.For example, one may consider to start constructing differential structures before constructing the Riemmannian structure.
- In our case we consider q derivations which are $q-$deformed derivations. These derivations are the classical vector fields acting on functions $f, g \in C^{\infty}(M)$.

$$X(fg) = X(f)g + fX(g) \tag{1}$$

Introduction

- With the $q-$deformed vector fields, we construct a connection, and by defining torsion freeness in a particular way, we seek to find conditions which defines Levi-Civita connection on the quantum $3-$sphere..
- Levi-Civita Connection is a connection that is metric and torsion free. In classical case, that usually means

$$\begin{aligned} [X, Y] &= \nabla_X Y - \nabla_Y X \quad \text{and} \\ Xg(Y, Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \end{aligned}$$

- Such a connection is characterised by Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(Y, [X, Z]) \\ -g(X, [Y, Z]) - g(Z, [Y, X]) \end{aligned}$$

Derivations

Let A be an associative(noncommutative) $\star-$algebra and denote $Der(A)$, the linear space of derivations on A. We have the following:

- $Der(A)$ is a Lie algebra when equipped with a Lie bracket.
- $Der(A)$ is a module over $Z(A)$,i.e., $Z(A)$ is the Center of A and $(f, X)(a) = fX(a), \forall f \in Z(A), X \in Der(A)$.
- For example, When $A = C^\infty(M)$, one has $Z(A) = C^\infty(M), Der(A) = \Gamma(M)$, the Lie algebra of vector fields.

Connection on modules

- Let M be an A- module, a connection is a map $\nabla : Der(A) \times M \longrightarrow M$ satisfying :
  (i) $\nabla_X(am) = a\nabla_X(m) + X(a)m$
  (ii) $\nabla_{fX}(m) = f\nabla_X(m)$
  (iii) $\nabla_{X+Y}(m) = \nabla_X(m) + \nabla_Y(m)$,
  $\forall X, Y \in Der(A), \forall a \in A, \forall m \in M, \forall f \in Z(A)$.
- For example the matrix algebra $A = M_n(\mathbb{C}), Z(A) = \mathbb{C}\mathbf{1}, Der(A) = Int(A) \approx \mathfrak{sl}_\mathfrak{n}$(traceless elements).
  For any connection ∇ on $M_n(\mathbb{C})$, considered as a right module on itself, one defines $\omega(X) = \nabla_X(\mathbf{1}), \forall X \in Der(M_n)$. Then ∇ is completely determined by the one form ω and its denoted by $\nabla^\omega : \forall X \in Der(M_n), \forall a \in M_n$,

$$\nabla^\omega_X a = [X, a] + \omega(X)a \qquad (2)$$

Metric on modules

- A sequilinear map $h : M \times M \longrightarrow A$ is such that

  $h(m_1 a_1, m_2 a_2) = a_1^\star h(m_1, m_2) a, \forall a_1, a_2 \in A, \forall m_1, m_2 \in M,$

  M considered as a right module.
- A connection is hermitean if

$$Xh(m_1, m_2) = h(\nabla_X m_1, m_2) + h(m_1, \nabla_X m_2),$$
$$\forall m_1, m_2 \in M, \forall X \in \mathfrak{g} \subset Der(A)$$

$q-$derivatives

We want connections on noncommutative algebra. Let A be a non commutative algebra, preferable a quantum group.

- When A is non-commutative, $Der(A)$ is not a module over $Z(A)$.
- Its more natural to have $q-$derivations on quantum groups.
- For example the space generated by elements $X_z, X_\pm$ taken as a tangent space of a quantum group S^3_q where $X_z = \dfrac{1-K^4}{1-q^{-2}}, X_+ = q^{\frac{1}{2}}EK, X_- = q^{-\frac{1}{2}}FK$, acting on elements of S^3_q as

$$\begin{aligned} X_\pm(ab) &= aX_\pm(b) + X_\pm(a)K^2(b) \quad \text{and} \\ X_z(ab) &= aX_z(b) + X_z(a)K^4(b) \end{aligned}$$

are actually $q-$derivations. Here E, F, K are generators of $U_q(su(2))$ acting on S^3_q.

$q-$derivations Cont.

- Actually they appear as

$$
\begin{aligned}
X_z(ab) &= aX_z(b) + X_z(a)b - (1-q^{-2})X_z(a)X_z(b) \\
X_\pm(ab) &= aX_\pm(b) + X_\pm(a)K^{-2}(b) - (1-q^{-2})X_\pm(a)K^{-2}X_z(b).
\end{aligned}
$$

- Quantum groups are usually considered as coordinate algebras realized after the deformation of the algebra of functions on a group. The resulting noncommutative group is usually what is referred to as a quantum group.

Noncommutative 3−Sphere S_q^3

- Looking closely at the noncommutative three sphere S_q^3 with its quantum tangent space TS_q^3.
- The quantum group S_q^3, as the coordinate algebra $O(SU_q(2))$ of Woronowicz is a Hopf $\star$- algebra generated by elements a and c satisfying the following relations:

$$\begin{aligned} ac &= qca \quad \text{and} \quad c^\star a^\star = qa^\star c^\star \\ ac^\star &= qc^\star a \quad \text{and} \quad ca^\star = qa^\star c \\ cc^\star &= c^\star c \quad \text{and} \quad a^\star a + c^\star c = aa^\star + q^2 cc^\star = 1. \end{aligned}$$

Quantum tangent space TS_q^3.

- The Quantum tangent space TS_q^3, a vector space over $\mathbb{C}$ spanned by elements $X_{\pm}, X_z$ satisfying the commutation relations:

$$X_{-}X_{+} - q^2 X_{+}X_{-} = X_z, \quad q^2 X_z X_{-} - q^{-2} X_{-} X_z = (1+q^2)X_{-}$$
$$\text{and} \quad q^2 X_{+} X_z - q^{-2} X_z X_{+} = (1+q^2)X_{+}.$$

$q-$deformed connection

- Taking M as a left S_q^3- module which has an action of $U_q(su(2))$, we define an affine connection as a map $\nabla : TS_q^3 \times M \longrightarrow M$, satisfying

$$\begin{aligned}
\nabla_X(m_1 + m_2) &= \nabla_X m_1 + \nabla_X m_2, \\
\nabla_{\lambda X+Y}(m_1) &= \lambda \nabla_X m_1 + \nabla_Y m_1. \quad \text{In} \quad \text{particular,} \\
\nabla_{X_\pm}(am) &= a\nabla_{X_\pm} m + X_\pm(a)K^2(m) \\
\nabla_{X_z}(am) &= a\nabla_{X_z} m + X_z(a)K^4(m), \quad \text{for} \quad \lambda \in \mathbb{C}, \\
& \quad X, Y \in TS_q^3, m_1, m_2 \in M \quad \text{and} \quad a \in S_q^3.
\end{aligned}$$

- This definition is motivated by the $q-$derivations:

$$\begin{aligned}
X_\pm(ab) &= aX_\pm(b) + X_\pm(a)K^2(b) \quad \text{and} \\
X_z(ab) &= aX_z(b) + X_z(a)K^4(b)
\end{aligned}$$

Remark

One immediately obtains:

$$\nabla_{X_z}(am) = X_z(a)(m - (1 - q^{-2})X_z(m)) + a\nabla_{X_z}(m) \quad \textit{and} \tag{3}$$

$$\nabla_{X_\pm}(am) = X_\pm(a)K^{-2}(m - (1 - q^{-2})X_z(m)) + a\nabla_\pm(m). \tag{4}$$

Definition

A hermitian form on the left A-module M is a map $h : M \times M \longrightarrow A$ such that

$$\begin{aligned} h(m_1 + m_2, m_3 &= h(m_1, m_3) + h(m_2, m_3) \\ h(am_1, m_2) &= ah(m_1, m_2) \\ h(m_1, m_2)^\star &= h(m_2, m_1) \end{aligned}$$

It is easy to see that these conditions in the above definition imply

$$h(m_1, m_2 + m_3) = h(m_1, m_2) + h(m_1, m_3)$$
$$h(m_1, am_2) = h(m_1, m_2)a^\star$$

Metric compatibility

Definition

Let M be a left S_q^3-module which has an action of $U_q(su(2))$, and assume that h is a hermitian form on M. We say that the affine connection ∇ on M is compatible with h if

$$X_{\pm}h(m_1, m_2) = -h(m_1, K^{-2}(\nabla_{\mp}m_2)) + h(\nabla_{\pm}m_1, K^{-2}(m_2))$$
$$X_z h(m_1, m_2) = -h(m_1, K^{-4}(\nabla_z m_2)) + h(\nabla_z m_1, K^{-4}(m_2)),$$

for all $m_1, m_2 \in M$.

We have written $\nabla_{X_{\pm}} = \nabla_{\pm}$ and $\nabla_{X_z} = \nabla_z$.

Proposition

Let $M = S_q^3$ and let $h(a, b) = ab^\star$. Defining

$$\nabla_\pm(a) = X_\pm(a) \quad and \quad \nabla_z(a) = X_z(a)$$

and extending it by linearity to TS_q^3, ∇ is an q - affine connection on M which is compatible with h.

The computation here:

$$\begin{aligned} X_\pm h(ab) &= X_\pm(ab^\star) \\ &= aX_\pm(b^\star) + X_\pm(a)K^2(b^\star) \\ &= a[S(X_\pm)^\star(b)]^\star + X_\pm(a)[S(K^2)^\star(b)]^\star \\ &= -a[K^{-2}X_\pm(b)]^\star + X_\pm(a)[K^{-2}(b)]^\star \\ &= -h(a, K^{-2}X_\pm(b)) + h(X_\pm(a), K^{-2}(b)) \\ &= -h(a, K^{-2}\nabla_\pm(b)) + h(\nabla_\pm(a), K^{-2}(b)) \end{aligned}$$

Proposition

The affine connection compatible with the hermitian form h is such that

$$[X_{\pm} h(m_1, m_2)]^{\star} = X_{\mp} h(m_1, m_2)$$
$$\textit{and} \quad [X_z h(m_1, m_2)]^{\star} = X_z h(m_1, m_2).$$

whenever $Kh(m_1, m_2) = h(K(m_1), K^{-1}(m_2))$.

Proof

To prove the proposition, one observes that

$$[X_{\pm} h(m_1, m_2)]^{\star} \quad = \quad -K^{-2} X_{\mp} h(m_2, m_1),$$

Proof continues

So that

$$-K^{-2}X_{\mp}h(m_2, m_1) = -h(K^{-2}(\nabla_{\mp}m_2), m_1) + h(K^{-2}(m_2), \nabla_{\pm}m_1)$$

By simply transposing $-K^{-2}$, one obtains the results.

- As defined by Woronowicz, If TS_q^3 is a quantum tangent space of a quantum group S_q^3, there exist a space of differential forms Ω^1 spanned by $\omega_+, \omega_-, \omega_z$ such that a differential form da is given by

$$da = \sum_{i=1}^{3} (X_i \triangleright a)\omega_i, \forall a \in S_q^3, \tag{5}$$

- Ω^1 is a bimodule.
  We fix the following:
- Define Christoffel symbols as $\nabla_a \omega_b = \Gamma^c_{ab}\omega_c = \Gamma^+_{ab}\omega_+ + \Gamma^-_{ab}\omega_- + \Gamma^z_{ab}\omega_z$, where $a, b = +, -, z$.
- And introduce a diagonal metric on Ω^1, writing $h(\omega_a, \omega_b) = \delta_{ab} h_a$. We also write $K^{-1}(\omega_a) = k_a \omega_a$.

- We see 15 metric equations with Christoffel symbols as

$$X_+ h_{ab} = -k_a^2 h_a K^{-2} (\Gamma^a_{-b})^\star + k_b^2 \Gamma^b_{+a} h_b$$
$$X_z h_{ab} = -k_a^4 h_a K^{-4} (\Gamma^a_{zb})^\star + k_b^4 \Gamma^b_{za} h_b$$

Particularly one has

$$\begin{aligned}
X_+ h_+ &= -k_+^{-2} h_+ K^{-2} (\Gamma^+_{-+})^\star + k_+^{-2} \Gamma^+_{++} h_+ \\
X_+ h_- &= -k_-^{-2} h_- K^{-2} (\Gamma^-_{--})^\star + k_-^{-2} \Gamma^-_{+-} h_- \\
X_+ h_z &= -k_z^{-2} h_z K^{-2} (\Gamma^z_{-z})^\star + k_z^{-2} \Gamma^z_{+z} h_z \\
X_+ h_{+-} &= -k_+^{-2} h_+ K^{-2} (\Gamma^+_{--})^\star + k_-^{-2} \Gamma^-_{++} h_- \\
X_+ h_{-+} &= -k_-^{-2} h_- K^{-2} (\Gamma^-_{-+})^\star + k_+^{-2} \Gamma^+_{+-} h_+ \\
X_+ h_{+z} &= -k_+^{-2} h_+ K^{-2} (\Gamma^+_{-z})^\star + k_z^{-2} \Gamma^z_{++} h_z \\
X_+ h_{-z} &= -k_-^{-2} h_- K^{-2} (\Gamma^-_{-z})^\star + k_z^{-2} \Gamma^z_{+-} h_z \\
X_+ h_{z+} &= -k_z^{-2} h_z K^{-2} (\Gamma^z_{-+})^\star + k_+^{-2} \Gamma^+_{+z} h_+ \\
X_+ h_{z-} &= -k_z^{-2} h_z K^{-2} (\Gamma^z_{--})^\star + k_-^{-2} \Gamma^-_{+z} h_-
\end{aligned}$$

$$
\begin{aligned}
X_z h_+ &= -k_+^{-4} h_+ K^{-4} (\Gamma_{z+}^{+})^\star + k_+^{-4} \Gamma_{z+}^{+} h_+ \\
X_z h_- &= -k_-^{-4} h_+ K^{-4} (\Gamma_{z-}^{-})^\star + k_-^{-4} \Gamma_{z-}^{-} h_- \\
X_z h_z &= -k_z^{-4} h_z K^{-4} (\Gamma_{zz}^{z})^\star + k_z^{-4} \Gamma_{zz}^{z} h_z \\
X_z h_{+-} &= -k_+^{-4} h_+ K^{-4} (\Gamma_{z-}^{+})^\star + k_-^{-4} \Gamma_{z-}^{-} h_- \\
X_z h_{+z} &= -k_+^{-4} h_+ K^{-4} (\Gamma_{zz}^{+})^\star + k_z^{-4} \Gamma_{z+}^{z} h_z \\
X_z h_{-z} &= -k_-^{-4} h_- K^{-4} (\Gamma_{zz}^{-})^\star + k_z^{-4} \Gamma_{z-}^{z} h_z
\end{aligned}
$$

We have the following observations:

- $$\begin{aligned} K(h_a)^\star &= h_a \\ h_c K(\Gamma^c_{ab})^\star &= K(\Gamma^c_{ab} h_c)^\star \end{aligned}$$

- $$X_z h_a = \frac{1(h_a) - K^4(h_a)}{1 - q^{-2}} \quad = \quad h_a - ha = 0$$

- $$X_z h_{ab} \quad = \quad \delta_{ab} X_z h_a = 0.$$

- $$X_+ h_{ab} \quad = \quad \begin{cases} X_+ h_a, & \text{if} \quad a = b \\ 0, & \quad\quad \text{otherwise} \end{cases}$$

Writing $\Gamma^{c}_{ab}h_c = \tilde{\Gamma}^{c}_{ab}$, the 15 metric equations can be summarised as:

$$\begin{aligned}
\tilde{\Gamma}^{a}_{+a} &= K^{-2}(\tilde{\Gamma}^{a}_{-a})^{\star} + k_a^{-2}X_{+}h_a \\
\tilde{\Gamma}^{b}_{+a} &= k_b^{-2}k_a^{2}K^{-2}(\tilde{\Gamma}^{a}_{-b})^{\star} \quad \text{and} \\
\tilde{\Gamma}^{a}_{za} &= K^{-4}(\tilde{\Gamma}^{a}_{za})^{\star} \\
\tilde{\Gamma}^{b}_{za} &= k_b^{-4}k_a^{4}K^{-4}(\tilde{\Gamma}^{a}_{zb})^{\star},
\end{aligned}$$

Where $k_a \neq k_b$.

Torsion freeness

Definition

Let M be a left S_q^3- module which has an action of $U_q(su(2))$ together with a soldering map $\varphi : TS_q^3 \longrightarrow M$. An affine connection is ∇ is called torsion free if

$$\begin{aligned} \varphi(X_z) &= \nabla_-\varphi(X_+) - q^2\nabla_+\varphi(X_-) \\ (1+q^2)\varphi(X_-) &= q^2\nabla_z\varphi(X_-) - q^{-2}\nabla_-\varphi(X_+) \\ (1+q^2)\varphi(X_+) &= q^2\nabla_+\varphi(X_z) - q^{-2}\nabla_z\varphi(X_+) \end{aligned}$$

We want $\varphi(X_+) = \omega_+$ and $\varphi(X_\pm) = \omega_\pm$.

Remark

The equations in the definition is motivated by the commutation relations of TS_q^3:

$$X_-X_+ - q^2 X_+X_- = X_z, \quad q^2 X_zX_- - q^{-2}X_-X_z = (1+q^2)X_-$$
$$\text{and} \quad q^2X_+X_z - q^{-2}X_zX_+ = (1+q^2)X_+.$$

The torsion free equations involving Christoffel symbols are

$$
\begin{aligned}
\nabla_{-}\omega_{+} - q^2\nabla_{+}\omega_{-} &= \omega_z \\
\Gamma^{+}_{-+} - q^2\Gamma^{+}_{+-} &= 0, \\
\Gamma^{-}_{-+} - q^2\Gamma^{-}_{+-} &= 0, \\
\Gamma^{z}_{-+} - q^2\Gamma^{z}_{+-} &= 1 \\
q^2\nabla_{z}\omega_{-} - q^{-2}\nabla_{-}\omega_{+} &= (1+q^2)\omega_{-} \\
q^2\Gamma^{+}_{z-} - q^{-2}\Gamma^{+}_{-z} &= 0, \\
q^2\Gamma^{-}_{z-} - q^{-2}\Gamma^{-}_{z-} &= 1+q^2, \\
q^2\Gamma^{z}_{z-} - q^{-2}\Gamma^{z}_{-z} &= 0 \\
q^2\nabla_{+}\omega_{z} - q^{-2}\nabla_{z}\omega_{+} &= (1+q^2)\omega_{+} \\
q^2\Gamma^{+}_{+z} - q^{-2}\Gamma^{+}_{z+} &= 1+q^2, \\
q^2\Gamma^{-}_{+z} - q^{-2}\Gamma^{-}_{z+} &= 0, \\
q^2\Gamma^{z}_{+z} - q^{-2}\Gamma^{z}_{z+} &= 0.
\end{aligned}
$$

Concluding remarks

We want construct metric and torsion free equations and see if we can find some solutions.
The work is on going.

Thank you for your Attention

Thank you!