Reordering in a multi-parametric family of algebras

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Objects of study

The main object treated is the unital associative free $\mathbb{C}$-algebra in arbitrary number of generators $\{S_j\}_{j\in J}$ and Q satisfying

$$S_j Q = \sigma_j(Q) S_j, \tag{1}$$

where σ_j is a Laurent polynomial. Denoting $S_{j_1} = S$, $S_{j_2} = T$, $\sigma_{j_1} = \sigma$ and $\tau_{j_2} = \tau$, we can write in three generators that

$$\begin{aligned} SQ &= \sigma(Q)S, \\ TQ &= \tau(Q)T. \end{aligned} \tag{2}$$

Writing $R = (dS - bT)/(ad - bc)$ and $J = (aT - cS)/(ad - bc)$, where a, b, c and d are complex numbers with $ad \neq bc$, we obtain

$$\begin{aligned} RQ &= \frac{ad\sigma(Q) - bc\tau(Q)}{ad - bc} R + \frac{bd\sigma(Q) - bd\tau(Q)}{ad - bc} J, \\ JQ &= \frac{ad\tau(Q) - bc\sigma(Q)}{ad - bc} J + \frac{ac\tau(Q) - ac\sigma(Q)}{ad - bc} R. \end{aligned} \tag{3}$$

Observe that relations (1) are recovered for $b = c = 0$.

Reordering

An arbitrary word (monomial) a in S and Q can be written as

$$a = S^{j_1} Q^{k_1} S^{j_2} Q^{k_2} \dots S^{j_r} Q^{k_r} \equiv \prod_{t=1}^{r} S^{j_t} Q^{k_t}. \tag{4}$$

a is called normal ordered if all the powers of Q stand to the left,

$$a = \sum_{j,k \in \mathbb{N}_0} A_{jk}(a) Q^j S^k. \tag{5}$$

The coefficients $A_{jk}(g)$ are called normal ordering coefficients of a.

Main result

If Q and $\{S_j\}_{j \in J}$ satisfy $S_j Q = \sigma_j(Q) S_j$, then for all positive integers k and r, and any polynomial F,

$$S_j^k F(Q) = F\left(\sigma_j^{\circ k}(Q)\right) S_j^k, \tag{6}$$

$$\left(S_j^k F(Q)\right)^r = \left(\prod_{t=1}^{r} F\left(\sigma_j^{\circ tk}(Q)\right)\right) S_j^{kr}, \tag{7}$$

and for all positive integers k_t and r, and any polynomials F_t,

$$\prod_{t=1}^{r} S_{j_t}^{k_t} F_t(Q) = \left(\prod_{t=1}^{r} F_t\left((\sigma_{j_t}^{\circ k_t} \circ \cdots \circ \sigma_{j_1}^{\circ k_1})(Q)\right)\right) \prod_{t=1}^{r} S_{j_t}^{k_t}, \tag{8}$$

where $\sigma^{\circ k}$ denotes the k-fold composition of σ with itself.

Main result

For positive integers k, N, r, and polynomial $F(Q) = \sum_{l=0}^{N} f_l Q^l$,

$$S_j^k F(Q) = \sum_{l=0}^{N} f_l \Big(\sigma_j^{\circ k}(Q)\Big)^l S_j^k, \tag{9}$$

$$\Big(S_j^k F(Q)\Big)^r = \sum_{(l_1,\ldots,l_r)\in\{0,\ldots,N\}^r} \left(\prod_{t=1}^{r} f_{l_t}\right)\left(\prod_{t=1}^{r}\Big(\sigma_j^{\circ tk}(Q)\Big)^{l_t}\right) S_j^{kr}, \tag{10}$$

and for all $k_t, N_t, r \in \mathbb{Z}_+$, and polynomials $F_t(Q) = \sum_{l_t=0}^{N_t} f_{l_t} Q^{l_t}$,

$$\begin{aligned}\prod_{t=1}^{r} S_{j_t}^{k_t} F_t(Q) = \sum_{(l_1,\ldots,l_r)\in I_1\times\ldots\times I_r} &\left(\prod_{t=1}^{r} f_{l_t}\right) \\ &\cdot \left(\prod_{t=1}^{r}\Big((\sigma_{j_t}^{\circ k_t}\circ\cdots\circ\sigma_{j_1}^{\circ k_1})(Q)\Big)^{l_t}\right)\prod_{t=1}^{r} S_{j_t}^{k_t},\end{aligned} \tag{11}$$

where $I_t = \{0,\ldots,N_t\}$ for some $t = 1,\ldots,r$,

Example: Three generators

If S, T and Q are elements of an algebra satisfying the relations

$$SQ = \sigma(Q)S \quad \text{and} \quad TQ = \tau(Q)T, \tag{12}$$

where σ and τ are polynomials, then for all positive integers J, k, l,

$$S^j T^k Q^l = \Big((\tau^{\circ k} \circ \sigma^{\circ j})(Q)\Big)^l S^j T^k, \tag{13}$$

$$\Big(S^j T^k Q^l\Big)^r = \left(\prod_{t=1}^{r} \Big((\tau^{\circ k} \circ \sigma^{\circ j})^{\circ t}(Q)\Big)^{l_t}\right)(S^j T^k)^r, \tag{14}$$

and for all positive integers j_t, k_t, l_t and r, where $t = 1, \ldots, r$,

$$\prod_{t=1}^{r} S^{j_t} T^{k_t} Q^{l_t} = \left(\prod_{t=1}^{r} \Big((\tau^{\circ k_t} \circ \sigma^{\circ j_t} \circ \cdots \circ \tau^{\circ k_1} \circ \sigma^{\circ j_1})(Q)\Big)^{l_t}\right) \prod_{t=1}^{r} S^{j_t} T^{k_t}.$$

Example: $\sigma_j(x) = c_j x^{q_j}$

Let $c_j \in \mathbb{C} \setminus \{0\}$, $q_j \in \mathbb{Z}$, and let σ_j be the polynomials

$$\sigma_j(x) = c_j x^{q_j}. \tag{15}$$

Then the general commutation relation becomes

$$S_j Q = c_j Q^{q_j} S_j. \tag{16}$$

and for positive integers r, the general reordering formula becomes

$$\prod_{t=1}^{r} S_{j_t}^{k_t} Q^{l_t} = \left(\prod_{n=1}^{r} c_{j_n}^{\{k_n\}_{q_{j_n}} \sum_{t=n}^{r} \left(\prod_{m=n+1}^{t} q_{j_m}^{k_m} \right) l_t} \right) Q^{\sum_{t=1}^{r} \left(\prod_{n=1}^{t} q_{j_n}^{k_n} \right) l_t} \prod_{t=1}^{r} S_{j_t}^{k_t}, \tag{17}$$

where $\{k\}_q$ for some complex number q denotes the q-number

$$\{k\}_q = \sum_{j=0}^{k-1} q^j = \frac{q^k - 1}{q - 1}. \tag{18}$$

General formula for nested commutators

What is $e^A e^B = e^{A+B}$ if $AB \neq BA$? (Baker–Campbell–Hausdorff)

- A function $f\colon \{1,\ldots,n\} \to \mathbb{R}$ is said to be *unimodal* if there exists some ν such that $f(1) \geq \cdots \geq f(\nu) \leq \cdots \leq f(n)$.
- A permutation of $[n] \equiv \{1,\ldots,n\}$ is a bijection $f : [n] \to [n]$.

Proposition

For all positive integers n, we have

$$\Big[x_n, \big[x_{n-1}, \ldots, [x_2, x_1]\ldots\big]\Big] = \sum_{\rho \in U_n} (-1)^{n-\rho^{-1}(1)} \prod_{\nu=1}^{n} x_{\rho(\nu)}, \qquad (19)$$

where U_n denotes the set of unimodal permutations of $\{1,\ldots,n\}$.

For $n = 2, 3$, we have

$$[x_2, x_1] = x_2 x_1 - x_1 x_2,$$
$$\big[x_3, [x_2, x_1]\big] = x_3 x_2 x_1 - x_3 x_1 x_2 - x_2 x_1 x_3 + x_1 x_2 x_3.$$

Nested commutator formulas for S_j, Q-elements

For any positive integers k_t, r_t, n, and any polynomials F_t, where $t = 1, \ldots, n$,

$$\left[S_{j_n}^{k_n} F_n(Q), \left[S_{j_{n-1}}^{k_{n-1}} F_{n-1}(Q), \ldots, \left[S_{j_2}^{k_2} F_2(Q), S_{j_1}^{k_1} F_1(Q)\right] \ldots\right]\right] =$$

$$= \sum_{\rho \in U_n} (-1)^{n-\rho^{-1}(1)} \left(\prod_{\nu=1}^{n} F_{\rho(\nu)}\Big((\sigma_{j_{\rho(\nu)}}^{\circ k_{\rho(\nu)}} \circ \cdots \circ \sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)}})(Q) \Big) \right) \prod_{\nu=1}^{n} S_{j_{\rho(\nu)}}^{k_{\rho(\nu)}}, \tag{20}$$

$$\left[\Big(S_{j_n}^{k_n} F_n(Q) \Big)^{r_n}, \ldots, \Big(S_{j_1}^{k_1} F_1(Q) \Big)^{r_1} \right] = \sum_{\rho \in U_n} (-1)^{n-\rho^{-1}(1)} \left(\prod_{\nu=1}^{n} \prod_{t=1}^{r_{\rho(\nu)}} \right.$$

$$\left. F_{\rho(\nu)}\Big((\sigma_{j_{\rho(\nu)}}^{\circ t k_{\rho(\nu)}} \circ \sigma_{j_{\rho(\nu-1)}}^{\circ k_{\rho(\nu-1)} r_{\rho(\nu-1)}} \circ \cdots \circ \sigma_{j_{\rho(1)}}^{\circ k_{\rho(1)} r_{\rho(1)}})(Q) \Big) \right) \prod_{\nu=1}^{n} S_{j_{\rho(\nu)}}^{k_{\rho(\nu)} r_{\rho(\nu)}}. \tag{21}$$

An operator representation

A concrete representation of the relation

$$S_j Q = \sigma_j(Q) S_j$$

is given by $S_j \mapsto \alpha_{\sigma_j}$ and $Q \mapsto Q_y$, where for any polynomial f,

$$\alpha_{\sigma_j}(f)(y) = f(\sigma_j(y)), \tag{22}$$

$$Q_y(f)(y) = yf(y), \tag{23}$$

Let $a, b, c, d \in \mathbb{C}$ with $ad \neq bc$, and let σ and τ be polynomials. Then the operators

$$R_{\sigma,\tau}(f)(y) = \frac{adf\left(\sigma(y)\right) - bcf\left(\tau(y)\right)}{ad - bc}, \tag{24}$$

$$J_{\sigma,\tau}(f)(y) = \frac{acf\left(\tau(y)\right) - acf\left(\sigma(y)\right)}{ad - bc}, \tag{25}$$

$$Q_y(f)(y) = yf(y), \tag{26}$$

acting on $\mathbb{C}[y]$ gives a representation of the R, J, Q-elements.

Thank you